QUALIFYING EXAM, Fall 2011

Algebraic Topology and Differential Geometry

NAME ____________________________ (PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER _______________ SIGNATURE _______________

Please do any 10 problems out of the following 20.

1. State and prove the Jordan-Brouwer Theorem. (If you would like to use some preliminary results, please state them clearly.)

2. Define the Hopf invariant. Assume the Hopf invariant is a homomorphism. Prove that \( h([\iota_{2n}^1, \iota_{2n}^1]) \) is non-zero, and use this to prove that \( \pi_{4n-1}(S^{2n}) \) contains \( \mathbb{Z} \).

3. Define Eilenberg-McLane space \( K(\pi, n) \). Prove that \( H_{n+1}(K(\pi, n); \mathbb{Z}) = 0 \) for \( n \geq 2 \) and an arbitrary abelian group \( \pi \).

4. Let \( f : S^n \times S^n \to S^{2n} \) be the quotient map collapsing \( S^n \vee S^n \) to a point. Show that \( f \) induces the zero map on all homotopy groups but \( f \) is not nullhomotopic.

5. Let \( M \) be a closed, orientable manifold of dimension \( 4k + 2 \). Show that the Euler characteristic of \( M \) is even.

6. Compute the homotopy groups \( \pi_q(CP^n) \) for \( q \leq 2n + 1 \).

7. State the Freudenthal Theorem. Let \( K, L \subset R^p \) be two finite simplicial complexes of dimensions \( k, l \) respectively. Let \( k + l + 1 < p \). Prove that the simplicial complexes \( K \) and \( L \) are not linked.

8. Define regular covering. Let \( X \) be the figure eight. Give a covering space \( p : Y \to X \) and a map \( f : Y \to Y \) so that \( pf = p \) and \( f \) is not a homeomorphism.

9. Let \( p : E \to B \) be a Serre fiber bundle, where \( B \) is a path connected space. Prove that for any two points \( x_0, x_1 \in B \) the fibers \( F_0 = p^{-1}(x_0) \) and \( F_1 = p^{-1}(x_1) \) are weak homotopy equivalent.

10. State the Lefschetz Fixed Point Theorem. Let

\[
    f : CP^{4k} \times \mathbb{R}P^{2n} \to CP^{4k} \times \mathbb{R}P^{2n}
\]

be a map. Prove that \( f \) always has a fixed point.
The following are differential geometry questions.

11. Let \( \mathbb{F}(u, v) = (u \cos v, u \sin v, f(u)) \) be a smooth parametrized surface in \( \mathbb{R}^3 \) with \( u > 0 \) and \( v \in (0, 2\pi) \). Compute

(a) the first fundamental form of this surface.

(b) the second fundamental form of this surface.

(c) the scalar curvature of this surface using (a) and (b).

12. Find a closed differential 2-form \( \omega \) on \( (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R} \) such that \( \omega \) is not exact. You need to justify your answer.

If you cannot do the above, you may do the following and get only half of the credits. Find a closed differential 1-form \( \omega_1 \) on \( (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R} \) such that \( \omega_1 \) is not exact. Again you need to justify your answer.

13. Consider differential 1-form \( \alpha = zdz - ydx \) on \( \mathbb{R}^3 \). Prove the following: if \( f(x, y, z) \in C^\infty(\mathbb{R}^3) \) satisfies that \( f\alpha \) is a closed 1-form, then \( f \) is identically zero. (Hint: use cylindrical coordinates.)

14. Let \( M_{3 \times 3} \) be the set of all \( 3 \times 3 \) real matrices and let \( SO(3) = \{ A \in M_{3 \times 3}, A^T A = I_{3 \times 3} \} \). Define the exponential map \( \exp: M_{3 \times 3} \rightarrow M_{3 \times 3} \) by

\[
\exp(B) = I + B + \frac{1}{2!} B^2 + \frac{1}{3!} B^3 + \frac{1}{4!} B^4 + \cdots
\]

(a) Prove that the series \( \exp(B) \) converges for any \( 3 \times 3 \) real matrix \( B \).

(b) Below you may assume that \( \exp \) is smooth and that \( \frac{\partial}{\partial s} \exp(B(s)) = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial}{\partial s} (B(s))^k \) for smooth matrix-valued function \( B(s) \) of \( s \in \mathbb{R} \).

(c) Show that \( \exp \) is an injective map on some neighborhood of \( 0_{3 \times 3} \).

(d) Prove that \( \exp(B) \in SO(3) \) when \( B \in M_{3 \times 3} \) satisfying \( B^T = -B \). Hence we can define \( \exp : \{ B \in M_{3 \times 3}, B^T = -B \} \rightarrow SO(3) \).

15. Let \( f, g : S^1 \rightarrow \mathbb{R}^2 \) be two smooth embeddings. Define set

\[
M = \{ (a, b, \bar{v}) \in S^1 \times S^1 \times \mathbb{R}^2, f(a) - g(b) = \bar{v} \}.
\]

(a) Show that \( M \) is a compact submanifold of \( S^1 \times S^1 \times \mathbb{R}^2 \).

(b) Let \( \pi : M \rightarrow \mathbb{R}^2 \) be the projection map \( \pi(a, b, \bar{v}) = \bar{v} \). Apply Sard's theorem to \( \pi \) to show that for almost all \( \bar{v} \in \mathbb{R}^2 \), \( f(S^1) \) is transversal to \( g(S^1) + \bar{v} \).

16. Consider the submanifold \( \iota : M \rightarrow \mathbb{R}^3 \) given by \( x^2 + y^2 - z^2 = 1 \).

(a) Show that the vector field

\[
X = \frac{xz}{1 + z^2} \frac{\partial}{\partial x} + \frac{yz}{1 + z^2} \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
\]
is tangent to $M$.

(b) Show that the two-form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ restricts to an area form on $M$, i.e., a two-form which never vanishes. (Hint: use cylindrical coordinates.)

(c) Does the flow of $X$ on $M$ preserve $\iota^*\omega$?

17. Let $\alpha$ be a smooth closed three-form on sphere $S^6$. Prove that the top form $\alpha \wedge \alpha$ vanishes at some point. (Note that the deRham cohomology $H^3_{deR}(S^6) = 0$)

18. Show that $S^1 \times S^2$ does not admit a metric of positive sectional curvature. Does your argument apply to $S^2 \times S^2$? Why? Note that one of Hopf's conjecture says that $S^2 \times S^2$ does not admit any metric of positive sectional curvature.

19. Let $(M^n, g)$ be a complete smooth Riemannian manifold. Let $F : M \to \mathbb{R}$ be a smooth function with a lower bound. Let $C^\infty([a, b], M)$ be the set of all smooth paths from $[a, b]$ to $M$ ($a < b$). Define the functional $\mathcal{L} : C^\infty([a, b], M) \to \mathbb{R}$ by

$$\mathcal{L}(\gamma) = \int_a^b (\gamma' \big|^2 + F(\gamma(t))) \, dt$$

where $\gamma' = \frac{d\gamma}{dt}$.

(a) Show that functional $\mathcal{L}$ has a lower bound.

(b) Does $\mathcal{L}$ have a upper bound? Justify your answer.

(c) Prove the following first variation formula of $\mathcal{L}$. Let $\gamma_\varepsilon, \varepsilon \in (-\varepsilon, \varepsilon)$, be a smooth variation of $\gamma_0 = \gamma$.

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \mathcal{L}(\gamma_\varepsilon) = 2 \langle T, S \rangle_{\gamma_0} + \int_a^b \langle -2 \nabla_T T + \nabla F, S \rangle \, dt$$

where $T = \gamma'$ is the tangent vector field and $S = \frac{d}{ds} \big|_{s=0} \gamma_s$ is the variational vector field along $\gamma$.

20. Use the moving frame method to compute the connection 1-form and the curvature 2-form of the metric $g = h(r)^2 dr^2 + f(r)^2 g_{H^{n-1}}$, where $h(r)$ and $f(r)$ are positive smooth functions and $g_{H^{n-1}}$ is the hyperbolic metric of the constant curvature $-1$. You may assume that $\{\omega^i\}_{i=1}^{n-1}$ is a local orthonormal 1-form frame of $(H^{n-1}, g_{H^{n-1}})$. 