QUALIFYING EXAM, Fall 2009
Algebraic Topology and Differential Geometry

NAME __________________________ (PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ___________ SIGNATURE __________________

Please do any 10 problems out of the following 20.

1. Define when a pair of topological spaces \((X, Y)\) is a Borsuk pair. Prove that a \(CW\)-pair \((X, Y)\) is a Borsuk pair (in the case when \(X, Y\) are finite complexes).

2. Define covering space. Let \(n \geq 2\). Prove that any map \(f: \mathbb{RP}^n \to S^1\) is homotopic to a constant map.

3. Let \(p: E \to B\) be a Serre fiber bundle, where \(B\) is a path connected space. Prove that for any two points \(x_0, x_1 \in B\) the fibers \(F_0 = p^{-1}(x_0)\) and \(F_1 = p^{-1}(x_1)\) are weak homotopy equivalent.

4. State the Lefschetz Fixed Point Theorem. Let

\[
f: \mathbb{CP}^{2009} \to \mathbb{CP}^{2009}
\]

be a map. Prove that \(f^2\) has a fixed point.

5. Define the Whitehead map \(w: S^{n+k-1} \to S^n \vee S^k\). Prove that the element \([w] \in \pi_{n+k-1}(S^n \vee S^k)\) is in the kernel of the suspension homomorphism

\[
\Sigma: \pi_{n+k-1}(S^n \times S^k) \to \pi_{n+k}(\Sigma(S^n \times S^k)).
\]

6. Let \(A: S^n \to S^n\) be the antipodal map, \(A: x \mapsto -x\), and \(\iota_n \in \pi_n(S^n)\) be the generator represented by the identity map \(S^n \to S^n\). Prove that the homotopy class \([A] \in \pi_n(S^n)\) is equal to

\[
[A] = \begin{cases} 
\iota_n, & \text{if } n \text{ is odd,} \\
-\iota_n, & \text{if } n \text{ is even.}
\end{cases}
\]

7. Consider the map

\[
g: S^{2n-2} \times S^3 \xrightarrow{\text{projection}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbb{CP}^{n}.
\]

Prove that \(g\) is not homotopic to a constant map.

8. Let \(X \subset S^n\) be homeomorphic to \(S^p\), \(1 \leq p \leq n-1\). Compute the homology groups \(\widetilde{H}_q(S^n \setminus X)\).

9. Let \(X\) be a finite simply-connected \(CW\)-complex with \(\widetilde{H}_n(X; \mathbb{Z}) = 0\) for all \(n\). Prove that \(X\) is contractible.

10. Let \(X\) be a path-connected space. Prove that the group \(H^1(X; \mathbb{Z})\) is free abelian group.
SOLUTIONS OF problems 1-10

1. Define when a pair of topological spaces \((X, Y)\) is a Borsuk pair. Prove that a \(CW\)-pair \((X, Y)\) is a Borsuk pair (in the case when \(X, Y\) are finite complexes).

**Solution:** First, recall that a pair (of topological spaces) \((X, A)\) a Borsuk pair, if for any map \(F : X \rightarrow Y\) a homotopy \(f_t : A \rightarrow Y\), \(0 \leq t \leq 1\), such that \(f_0 = F|_A\) may be extended up to homotopy \(F_t : X \rightarrow Y\), \(0 \leq t \leq 1\), such that \(F_t|_A = f_t\) and \(F_0 = F\).

We are given a map \(\Phi : A \times I \rightarrow Y\) (a homotopy \(f_t\)) and a map \(F : X \times \{0\} \rightarrow Y\), such that \(F|_{A \times \{0\}} = \Phi|_{A \times \{0\}}\). To extend a homotopy \(f_t\) up to homotopy \(F_t\) is the same as to construct a map \(F' : X \times I \rightarrow Y\) such that \(F'|_{A \times I} = \Phi\). We construct \(F'\) by induction on dimension of cells of \(X \setminus A\). The first step is to extend \(\Phi\) to the space \((A \cup X(0)) \times I\) as follows:

\[
F'(x, t) = \begin{cases} 
F(x), & \text{if } x \text{ is a 0-cell from } X, \ x \notin A, \\
\Phi(x, t), & \text{if } x \in A.
\end{cases}
\]

Now assume that \(F'\) is already defined on \((A \cup X(n)) \times I\). Let \(e^{n+1}\) be a \((n+1)\)-cell, \(e^{n+1} \subset X \setminus A\). By induction, the map \(F'\) is already given on the cylinder \(e^{n+1} = e^{n+1} \times I\) since the boundary \(\partial e^{n+1} \cup e^{n+1} \subset X^n\). Let \(g : D^{n+1} \rightarrow X(\{n+1\})\) be a characteristic map corresponding to the cell \(e^{n+1}\). We have to define an extension of \(F'\) from the side \(g(S^n) \times I\) and the bottom base \(g(D^{n+1}) \times \{0\}\) to the cylinder \(g(D^{n+1}) \times I\). By definition of \(CW\)-complex, it is the same as to construct an extension of the map

\[
\psi = F' \circ g : (S^n \times I) \cup (D^{n+1} \times \{0\}) \rightarrow Y
\]

to a map of the cylinder \(\psi' : D^{n+1} \times I \rightarrow Y\). Let

\[
\eta : D^{n+1} \times I \rightarrow (S^n \times I) \cup (D^{n+1} \times \{0\})
\]

be a projection map of the cylinder \(D^{n+1} \times I\) from a point \(s\) which is near and a bit above of the top side \(D^{n+1} \times \{1\}\) of the cylinder \(D^{n+1} \times I\), see the picture below:

The map \(\eta\) is an identical map on \((S^n \times I) \cup (D^{n+1} \times \{0\})\). We define an extension \(\psi'\) as follows:

\[
\psi' : D^{n+1} \times I \xrightarrow{\eta} (S^n \times I) \cup (D^{n+1} \times \{0\}) \xrightarrow{\psi} Y.
\]

This procedure may be carried out independently for all \((n+1)\)-cells of \(X\), so we obtain an extension

\[
F' : (A \cup X(\{n+1\})) \times I \rightarrow Y.
\]

Thus, going from the skeleton \(X^{(n)}\) to the skeleton \(X^{(n+1)}\), we construct an extension \(F' : X \times I \rightarrow Y\) of the map \(\Phi : A \times I \rightarrow Y\).

We should emphasize that if \(X\) is an infinite-dimensional complex, then our construction consists of infinite number of steps; in that case the axiom (W) implies that \(F'\) is a continuous map. ⊓⊔
2. Define covering space. Let \( n \geq 2 \). Prove that any map \( f : \mathbb{RP}^n \to S^1 \) is homotopic to a constant map.

**Solution:** A path-connected space \( T \) is a covering space over a path-connected space \( X \), if there is a map \( p : T \to X \) such that for any point \( x \in X \) there exists a path-connected neighbourhood \( U \subset X \) such that \( p^{-1}(U) \) is homeomorphic to \( U \times \Gamma \) (where \( \Gamma \) is a discrete set), furthermore the following diagram commutes

\[
\begin{array}{ccc}
p^{-1}(U) & \xrightarrow{\cong} & U \times \Gamma \\
\downarrow{p} & & \downarrow{pr} \\
U & & \\
\end{array}
\]  

(1)

The neighbourhood \( U \) from the above definition is called elementary neighborhood. \( \square \)

Let \( f : \mathbb{RP}^n \to S^1 \) be a map. We recall that \( \pi_1 \mathbb{RP}^n \cong \mathbb{Z}/2 \), and \( \pi_1 S^1 \cong \mathbb{Z} \). Thus the induced homomorphism \( f_* : \pi_1 \mathbb{RP}^n \to \pi_1 S^1 \) is trivial. Consider the universal covering \( \mathbb{R} \xrightarrow{p} S^1 \). Let \( x_0 \in S^1 \) be a base point. Recall the following result:

**Theorem A.** Let \( p : T \to X \) be a covering space, and \( Z \) be a path-connected space, \( x_0 \in X \), \( \bar{x}_0 \in T \), \( p(\bar{x}_0) = x_0 \). Given a map \( f : (Z, z_0) \to (X, x_0) \) there exists a lifting \( \tilde{f} : (Z, z_0) \to (T, \bar{x}_0) \) if and only if \( f_*([x_0]) \subset p_*([T, \bar{x}_0]) \).

Since the homomorphism \( f_* : \pi_1 \mathbb{RP}^n \to \pi_1 S^1 \) is trivial, according to Theorem A, there exists a lift \( \tilde{f} : \mathbb{RP}^n \to \mathbb{R} \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\tilde{f}} & \mathbb{RP}^n \\
\downarrow{\bar{f}} & & \downarrow{p} \\
\mathbb{RP}^n & \xrightarrow{f} & S^1 \\
\end{array}
\]

Now \( \tilde{f} : \mathbb{RP}^n \to \mathbb{R} \) is null-homotopic since \( \mathbb{R} \) is contractible. Thus the composition

\[ f = p \circ \tilde{f} : \mathbb{RP}^n \to S^1 \]

is also null-homotopic. \( \square \)

3. Let \( p : E \to B \) be a Serre fiber bundle, where \( B \) is a path connected space. Prove that for any two points \( x_0, x_1 \in B \) the fibers \( F_0 = p^{-1}(x_0) \) and \( F_1 = p^{-1}(x_1) \) are weak homotopy equivalent.

**Solution:** Let \( s : I \to B \) be a path connecting \( x_0 \) and \( x_1 \). We have to define one-to-one correspondence \( \varphi_K : [K, F_0] \to [K, F_1] \) for any CW-complex \( K \).

Let \( h_0 : K \to F_0 \) be a map. Denote \( i_0 : F_0 \to E \) the inclusion map. We have the map:

\[ \tilde{f} : K \xrightarrow{h_0} F_0 \xrightarrow{i_0} E. \]

Consider also the homotopy \( F : K \times I \to B \), where \( G(x, t) = s(t) \). By the CHF there exists a covering homotopy \( \tilde{F} : K \times I \to E \) of the map \( f \) such that \( p \circ \tilde{F} = F \), in particular, \( \tilde{F}(K \times \{t\}) \subset p^{-1}(s(t)) \), and \( \tilde{F}(K \times \{1\}) \subset F_1 \).

We define \( \varphi_K(h_0 : K \to F_0) = (h_1 : K \to F_1) \), where \( h_1 = \tilde{F}|_{K \times \{1\}} \). We should show that the map \( \varphi_K \) is well-defined.
Let \( s' \) be a different path connecting \( x_0 \) and \( x_1 \), and \( \tilde{F} : K \to E, \tilde{F}' : K \times I \to B, h' : K \to F_1 \) be corresponding maps and homotopies determined by \( s' \). Assume that \( s \) and \( s' \) are homotopic, and let \( S : I \times I \to B \) be a corresponding homotopy. Denote by \( T : I \times I \to B \) a map defined by \( T(t_1, t_2) = S(t_2, t_1) \), see Fig. (a) We are going to use the relative version of the CHP for the pair \( Z' \subset Z \) where \( Z = K \times I \) and \( Z' = K \times \{0, 1\} \).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
(K \times \{0, 1\}) \times I & \longrightarrow & (K \times I) \times I \\
\uparrow & & \downarrow G \\
(K \times \{0, 1\}) \times \{0\} & \longrightarrow & (K \times I) \times \{0\}
\end{array}
\]

(2)

Here the map \( g : (K \times I) \times \{0\} \to B \) sends everything to \( x_0 \), and \( \tilde{g} : K \times I \to E \) defined by \( \tilde{g}(k, t_1) = f(k) \) (see above). The homotopy \( G : (K \times I) \times I \to B \) is defined by the formula: \( G(k, t_1, t_2) = T(t_1, t_2) \). The map \( \tilde{G}' : (K \times \{0, 1\}) \times I \to E \) is defined by the homotopies \( F \) and \( F' \):

\[
\tilde{G}'|_{K \times \{0\} \times I} = F, \quad \tilde{G}'|_{K \times \{1\} \times I} = F'.
\]

The relative version of the CHP implies that there exists \( \tilde{G} : K \times I \to E \) covering \( G \) and \( \tilde{G}' \) as it is shown at in (2).

The map \( (k, t) \to \tilde{G}(k, t, 1) \) maps \( K \times I \) to \( F_1 \); this is the homotopy connecting \( h_1 \) and \( h' \), see Fig. (b). Thus a path \( s : I \to B \) defines a map \( \varphi_K(s) : [K, F_0] \to [K, F_1], F_0 = p^{-1}(s(0)), F_1 = p^{-1}(s(1)) \), which does depend only of the homotopy class of \( s \). Clearly the map \( \varphi_K \) is natural with respect to \( K \); note also that if \( s \) is a constant path, then \( \varphi_K = Id_{F_0} \). Moreover, if a composition of paths \( s_2 \cdot s_1 \) (i.e. \( s_1(1) = s_2(0) \)) gives a map \( \varphi_K(s_2 \cdot s_1) = \varphi_K(s_2) \circ \varphi_K(s_1) \). In particular, the map \( \varphi_K(s^{-1}) \) is inverse to \( \varphi_K(s) \); it implies that \( \varphi_K(s) \) is one-to-one.

4. State the Lefschetz Fixed Point Theorem. Let 

\[
f : \mathbb{CP}^{2009} \to \mathbb{CP}^{2009}
\]

be a map. Prove that \( f^2 \) has a fixed point.

**Solution:** Let \( A \) be a finitely generated abelian group. Denote \( F(A) \) the free part of \( A \), so that \( A = F(A) \oplus T(A) \), where \( T(A) \) is a maximum torsion subgroup of \( A \). Let \( \varphi : A \to A \) be an endomorphism of \( A \). We define \( F(\varphi) : F(A) \to F(A) \) by composition:

\[
F(\varphi) : F(A) \xrightarrow{\text{inclusion}} A \xrightarrow{\varphi} A \xrightarrow{\text{projection}} F(A).
\]
We define \( \text{Tr}(\varphi) = \text{Tr}(F(\varphi)) \). Now let \( A = \{ A_q \}_{q \geq 0} \) be a finitely generated graded abelian group, i.e. each group \( A_q \) is finitely generated. A homomorphism \( \Phi : A \to B \) of two graded abelian groups is a collection of homomorphisms \( \{ \varphi_q : A_q \to B_{q-k} \} \) (the number \( k \) is the degree of \( \Phi \)).

Now let \( A = \{ A_q \}_{q \geq 0} \) be a finitely generated graded abelian group, and let \( \Phi = \{ \varphi_q \} : A \to A \) be an endomorphism of degree zero. We assume that \( \Phi(A_q) = 0 \) for \( q \geq n \) (for some \( n \)). We define the \textit{Lefschetz number} \( \text{Lef}(\Phi) \) of the endomorphism \( \Phi \) by the formula:

\[
\text{Lef}(\Phi) = \sum_{q \geq 0} (-1)^q \text{Tr}(\varphi_q).
\]

**Lefschetz Fixed Point Theorem.** Let \( X \) be a finite CW-complex, \( f : X \to X \) be a map such that \( \text{Lef}(f) = 0 \). Then \( f \) has a fixed point, i.e. such point \( x_0 \in X \) that \( f(x_0) = x_0 \).

Now we consider a map

\[
f : \mathbb{C}P^{2k-1} \to \mathbb{C}P^{2k-1}, \quad \text{where} \quad k \geq 1.
\]

Let \( x \in H^2(\mathbb{C}P^{2k-1}; \mathbb{Z}) \cong \mathbb{Z} \) be a generator. We know that \( H^*(\mathbb{C}P^{2k-1}; \mathbb{Z}) \cong \mathbb{Z}[x]/x^{2k} \). Then \( f^*(x) = \lambda \cdot x \), where \( \lambda \in \mathbb{Z} \). The map \( f \) induces the ring homomorphism

\[
f^* : H^*(\mathbb{C}P^{2k-1}; \mathbb{Z}) \to H^*(\mathbb{C}P^{2k-1}; \mathbb{Z}).
\]

We have: \( f^*(x^q) = \lambda^q x^q \) for \( q = 1, 2, \ldots, 2k - 1 \). We notice that the Universal coefficient formula gives that the homomorphism

\[
f_* : H_{2q}(\mathbb{C}P^{2k-1}; \mathbb{Z}) \to H_{2q}(\mathbb{C}P^{2k-1}; \mathbb{Z})
\]

is also the multiplication by \( \lambda^q \). Thus we have the following homomorphisms in the homology groups:

\[
\begin{array}{cccc}
q &=& 0 & H_0(\mathbb{C}P^{2k-1}; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{f_*} \mathbb{Z} \cong H_0(\mathbb{C}P^{2k-1}; \mathbb{Z}) \text{ since } \mathbb{C}P^{2k} \text{ is connected}, \\
q &=& 2 & H_2(\mathbb{C}P^{2k-1}; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{-\lambda} \mathbb{Z} \cong H_2(\mathbb{C}P^{2k-1}; \mathbb{Z}) \\
& & & \cdots \\
q &=& 4k-2 & H_{4k-2}(\mathbb{C}P^{2k-1}; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\lambda^{2k-1}} \mathbb{Z} \cong H_{4k-2}(\mathbb{C}P^{2k-1}; \mathbb{Z})
\end{array}
\]

Thus we have the trace

\[
\text{Lef}(f) = 1 + \lambda + \lambda^2 + \ldots + \lambda^{2k-1} = \frac{1 - \lambda^{2k}}{1 - \lambda}.
\]

Then it is easy to compute the trace \( \text{Lef}(f^2) \):

\[
\text{Lef}(f^2) = 1 + \lambda^2 + \lambda^4 + \ldots + \lambda^{4k-2} = \frac{1 - \lambda^{4k}}{1 - \lambda^2}.
\]

Assume that \( \lambda = \pm 1 \). Then \( \text{Lef}(f^2) = 1 + \lambda^2 + \lambda^4 + \ldots + \lambda^{4k-2} \neq 0 \). On the other hand, the only possibility for \( \text{Lef}(f^2) = 0 \) is \( \lambda^k = 1 \) or \( \lambda = \pm 1 \). Thus \( \text{Lef}(f^2) \neq 0 \) for any map \( f \), and the map \( f^2 \) has a fixed point. \( \square \)

**5.** Define the Whitehead map \( w : S^{n+k} \to S^n \vee S^k \). Prove that the element \([w] \in \pi_{n+k-1}(S^n \vee S^k)\) is in the kernel of the suspension homomorphism

\[
\Sigma : \pi_{n+k-1}(S^n \times S^k) \to \pi_{n+k}(\Sigma(S^n \times S^k)).
\]

**Solution:** Consider the product \( S^n \times S^k \) as a CW-complex. We choose a cell decomposition of \( S^n \times S^k \) into four cells of dimensions 0, n, k, n + k. The first three cells give us the wedge \( S^n \vee S^k \subset S^n \times S^k \).
The last cell $e^{n+k} \subset S^n \times S^k$ has the attaching map $w : S^{n+k-1} \to S^n \vee S^k$. This attaching map is called the Whitehead map. It is convenient to have a particular construction of the map $w$.

We can think about the sphere $S^{n+k-1}$ as a boundary of the unit disk $D^{n+k} \subset \mathbb{R}^{n+k}$. Thus a point $x \in S^{n+k-1}$ has coordinates $(x_1, \ldots, x_{n+k})$, where $x_1^2 + \cdots + x_{n+k}^2 = 1$. We define

$$U = \{(x_1, \ldots, x_{n+k}) \in S^{n+k-1} | x_1^2 + \cdots + x_n^2 \leq 1/2\},$$

$$V = \{(x_1, \ldots, x_{n+k}) \in S^{n+k-1} | x_{n+1}^2 + \cdots + x_{n+k}^2 \leq 1/2\}.$$

The map $w : S^{n+k-1} \to S^n \vee S^k$ is defined as follows. First we construct the maps $\varphi_U : U \to S^n \vee S^k$ and $\varphi_V : V \to S^n \vee S^k$ as the compositions:

$$\varphi_U : U \xrightarrow{\cong} D^n \times S^{k-1} \xrightarrow{pr} D^n \to D^n / S^{n-1} \xrightarrow{\cong} S^n \to S^n \vee S^k,$$

$$\varphi_V : V \xrightarrow{\cong} S^{n-1} \times D^k \xrightarrow{pr} D^k \to D^k / S^{k-1} \xrightarrow{\cong} S^k \to S^n \vee S^k.$$

Clearly we have that

$$\varphi_U|_{S^{n-1} \times S^{k-1}} = * = \varphi_V|_{S^{n-1} \times S^{k-1}}$$

and hence the maps $\varphi_U, \varphi_V$ define the map $w : S^{n+k-1} \to S^n \vee S^k$.

**Lemma A.** The Whitehead element $[w] \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \to \pi_{n+k}(\Sigma(S^n \times S^k)).$$

**Proof.** First, we state (and prove) several facts. For the exam, it is enough to state Lemmas B, C, and prove Lemma A using those results.

**Lemma B.** The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in a kernel of each of the following homomorphisms:

1. $\iota : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k-1}(S^n \times S^k)$,
2. $\text{pr}^{(n)}_* : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k-1}(S^n)$,
3. $\text{pr}^{(k)}_* : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k-1}(S^k)$.

**Proof of Lemma B.** The exact sequence

$$\to \pi_{n+k}(S^n \times S^k) \xrightarrow{i_*} \pi_{n+k-1}(S^n \vee S^k) \xrightarrow{\text{pr}^{(n)}_*} \pi_{n+k-1}(S^n \times S^k)$$

implies that $w \in \text{Ker} \ i_*$ since $w = \partial(i)$.

The commutative diagram

$$\begin{array}{ccc}
\pi_{n+k-1}(S^n \vee S^k) & \xrightarrow{\iota_*} & \pi_{n+k-1}(S^n \times S^k) \\
\downarrow \text{pr}^{(n)}_* & & \downarrow \text{pr}^{(n)}_* \\
\pi_{n+k-1}(S^n) & & \pi_{n+k-1}(S^n \times S^k)
\end{array}$$

(where $\text{pr} : S^n \times S^k \to S^n$ is a map collapsing $S^k$ to the base point) implies that $w \in \text{Ker} \ \text{pr}^{(n)}_*$ and similarly $w \in \text{Ker} \ \text{pr}^{(k)}_*$.

**Proof of Lemma A.** Consider the suspension homomorphism

$$\Sigma : \pi_q(S^n \vee S^k) \to \pi_{q+1}(\Sigma(S^n \vee S^k)).$$
Consider the commutative diagram:

\[
\begin{array}{cccccc}
\pi_{n+k-1}(S^n) & \xrightarrow{pr_n^{(n)}} & \pi_{n+k-1}(S^n \vee S^k) & \xrightarrow{pr_k^{(n)}} & \pi_{n+k-1}(S^k) \\
\Sigma & \downarrow & \Sigma & \downarrow & \Sigma \\
\pi_{n+k}(S^{n+1}) & \xrightarrow{\Sigma (pr_n^{(n)})} & \pi_{n+k}(S^{n} \vee S^k) & \xrightarrow{\Sigma (pr_k^{(n)})} & \pi_{n+k}(S^{k+1})
\end{array}
\]

where \( pr \) denote the collapsing maps. By Lemma B \( w \in \text{Ker } pr_n^{(n)} \), \( w \in \text{Ker } pr_k^{(n)} \). Notice that \( \Sigma(S^n \vee S^k) \sim S^{n+1} \vee S^{k+1} \). We need the following fact.

**Lemma C.** There is an isomorphism

\[
\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})
\]

**Proof of Lemma C.** Consider the long exact sequence for the pair \((S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1})\):

\[
\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \xrightarrow{\partial} \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{\iota_*} \pi_{n+k}(S^{n+1} \times S^{k+1}) \xrightarrow{\delta_*} \pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1})
\]

We notice that the \((n+k+1)\)-skeleton of the product \(S^{n+1} \times S^{k+1}\) is the wedge \(S^{n+1} \vee S^{k+1}\).

Thus any map \(D^{k+1+1} \rightarrow S^{n+1} \times S^{k+1}\) may be deformed to the subcomplex \(S^{n+1} \vee S^{k+1}\). Thus \(\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0\). The same argument gives that

\[
\pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0.
\]

Thus the long exact sequence (4) gives the isomorphism:

\[
i_* : \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{\cong} \pi_{n+k}(S^{n+1} \times S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1}). \quad \square
\]

To complete the proof of Lemma A we notice that Lemma C and the diagram (3) imply that \( w \in \text{Ker } \Sigma \). \( \square \)

6. Let \( A : S^n \rightarrow S^n \) be the antipodal map, \( A : x \mapsto -x \), \( i_n \in \pi_n(S^n) \) be the generator represented by the identity map \( S^n \rightarrow S^n \). Prove that the homotopy class \([A] \in \pi_n(S^n)\) is equal to

\[
[A] = \begin{cases}
i_n, & \text{if } n \text{ is odd}, \\
-i_n, & \text{if } n \text{ is even}.
\end{cases}
\]

**Solution:** Let \( S^n \) be given as \( x_1^2 + \cdots + x_n^2 = 1 \) in \( \mathbb{R}^{n+1} \). Let \( n = 2k-1 \), then \( n-1 = 2k \). The antipodal map is given as \( A : (x_1, \ldots, x_{2k}) \mapsto (-x_1, \ldots, -x_{2k}) \). Let \( 0 \leq \theta \leq \pi \). Consider the homotopy

\[
A_\theta = \begin{pmatrix}
\cos \theta & \sin \theta & \cdots & 0 & 0 \\
-\sin \theta & \cos \theta & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cos \theta & \sin \theta \\
0 & 0 & \cdots & -\sin \theta & \cos \theta
\end{pmatrix}
\]

Then \( A_0 = \text{Id}, A_\pi = A \). Thus \([A] = i_n\) if \( n \) is odd.

The case \( n = 2k \) is similar. We consider the homotopy

\[
A_\theta = \begin{pmatrix}
\cos \theta & \sin \theta & \cdots & 0 & 0 & 0 \\
-\sin \theta & \cos \theta & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cos \theta & \sin \theta & 0 \\
0 & 0 & \cdots & -\sin \theta & \cos \theta & 0 \\
0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}
\]

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Then $A_0 = -t_n$, and $A_n = A$.

7. Consider the map

$$g : S^{2n-2} \times S^3 \xrightarrow{\text{projection}} (S^{2n-2} \times S^3) / (S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbb{C}P^n.$$ 

Prove that $g$ is not homotopic to a constant map.

**Solution:** We put together the map $g$ in the following diagram:

$$(S^{2n-2} \times S^3) / (S^{2n-2} \vee S^3) \xrightarrow{\text{Hopf}} S^{2n+1}$$

By construction, we obtain that the top $(2n + 1)$-cell of the product $S^{2n-2} \times S^3$ maps to the top $(2n + 1)$-cell of $S^{2n+1}$. In particular, it means that the induced homomorphism in homology

$$g_* : H_{2n+1}(S^{2n-2} \times S^3; \mathbb{Z}) \rightarrow H_{2n+1}(S^{2n+1}; \mathbb{Z})$$

is isomorphism. Assume $g$ is homotopic to a constant map. Then since $S^{2n+1} \xrightarrow{\text{Hopf}} \mathbb{C}P^n$ is a fiber bundle, a null-homotopy for $g$ can be lifted to a homotopy of $\tilde{g}$ to a map $\bar{g}$,

$$\bar{g} : S^{2n-2} \times S^3 \rightarrow S^{2n+1}$$

such that the composition Hopf $\circ \bar{g}$ maps $S^{2n-2} \times S^3$ to some point $z_0 \in \mathbb{C}P^n$. It means that the map $\bar{g}$ factors through the fiber:

![Diagram](image)

On the other hand, $\tilde{g}_* = \bar{g}_*$ in homology and the diagram (6) gives a commutative diagram:

![Diagram](image)

Clearly $\bar{g}_* = 0$. Thus the homomorphism $\bar{g}_* = \bar{g}_*$ must be trivial as well. Contradiction.

$\square$
8. Let $X \subset S^n$ be homeomorphic to $S^p$, $1 \leq p \leq n - 1$. Compute the homology groups $\tilde{H}_q(S^n \setminus X)$.

**Solution:** First we prove a technical result.

**Lemma A.** Let $K \subset S^n$ be homeomorphic to the cube $I^k$, $0 \leq k \leq n$. Then

$$\tilde{H}_q(S^n \setminus K) = 0 \quad \text{for all } q \geq 0.$$ 

**Proof.** Induction on $k$. The case $k = 0$ is obvious. Assume that the statement holds for all $0 \leq k \leq m - 1$, and let $K$ is homeomorphic to $I^m$. We choose a decomposition $K = L \times I$, where $L$ is homeomorphic to $I^{m-1}$. Let $K_1 = L \times [0, \frac{1}{2}]$, and $K_2 = L \times \left[\frac{1}{2}, 1\right]$. Then $K_1 \cap K_2 = L \times \left\{\frac{1}{2}\right\} \cong I^{m-1}$. By induction,

$$\tilde{H}_q(S^n \setminus K_1 \cap K_2) = 0 \quad \text{for all } q \geq 0.$$ 

We notice that the sets $S^n \setminus K_1$, $S^n \setminus K_2$ are both open in $S^n$. Thus we can use the Mayer-Vietoris exact sequence

$$\cdots \to \tilde{H}_q(S^n \setminus (K_1 \cup K_2)) \to \tilde{H}_q(S^n \setminus K_1) \oplus \tilde{H}_q(S^n \setminus K_2) \to \tilde{H}_q(S^n \setminus (K_1 \cap K_2)) \to \cdots$$

Thus we have that

$$\tilde{H}_q(S^n \setminus (K_1 \cup K_2)) \cong \tilde{H}_q(S^n \setminus K_1) \oplus \tilde{H}_q(S^n \setminus K_2).$$

Assume that $\tilde{H}_q(S^n \setminus (K_1 \cup K_2)) \neq 0$, and $z_0 \in \tilde{H}_q(S^n \setminus (K_1 \cup K_2))$, $z_0 \neq 0$. Then $z_0 = (z_0', z_0'')$, thus there exists $z_1 \neq 0$ in the group $\tilde{H}_q(S^n \setminus K_1)$ or $\tilde{H}_q(S^n \setminus K_2)$. Let, say, $z_1 \in \tilde{H}_q(S^n \setminus K_1)$, $z_1 \neq 0$. Then we repeat the argument for $K_1$, and obtain the sequence

$$K \supset K^{(1)} \supset K^{(2)} \supset \cdots$$

such that

(1) $K^{(s)}$ is homeomorphic to $I^m$,

(2) the inclusion $i_s : S^n \setminus K \subset S^n \setminus K^{(s)}$ takes the element $z$ to a nonzero element $z_s \in \tilde{H}_q(S^n \setminus K^{(s)})$,

(3) the intersection $\bigcap_{s} K^{(s)}$ is homeomorphic to $I^{m-1}$.

We have that any compact subset $C$ of $S^n \setminus \bigcap_{s} K^{(s)}$ lies in $S^n \setminus K^{(s)}$ for some $s$, we obtain that

$$C_q(S^n \setminus \bigcap_{s} K^{(s)}) = \lim_{s} C_q(S^n \setminus K^{(s)}),$$

and, respectively,

$$\tilde{H}_q(S^n \setminus \bigcap_{s} K^{(s)}) = \lim_{s} \tilde{H}_q(S^n \setminus K^{(s)}).$$

By construction, there exists an element $z_\infty \in \tilde{H}_q(S^n \setminus \bigcap_{s} K^{(s)})$, $z_\infty \neq 0$. Contradiction to the inductive assumption. \qed

**Theorem.** Let $S^k \subset S^n$, $0 \leq k \leq n - 1$. Then

$$\tilde{H}_q(S^n \setminus S^k) \cong \begin{cases} 
Z, & \text{if } q = n - k - 1, \\
0, & \text{if } q \neq n - k - 1.
\end{cases} \quad (7)$$

**Proof.** Induction on $k$. If $k = 0$, then $S^n \setminus S^0$ is homotopy equivalent to $S^{n-1}$. Thus the formula (7) holds for $k = 0$. Let $k > 1$, then $S^k = D^k_+ \cup D^k_-$, where $D^k_+$, $D^k_-$ are the south and northern hemispheres of $S^k$. Clearly $D^k_+ \cap D^k_- = S^{k-1}$. Notice that the sets $S^n \setminus D^k_{\pm}$ are open in $S^n$, we can use the Mayer-Vietoris exact sequence:

$$\cdots \to \tilde{H}_{q+1}(S^n \setminus D^k_+) \oplus \tilde{H}_{q+1}(S^n \setminus D^k_-) \to \tilde{H}_{q+1}(S^n \setminus D^k_+ \cap D^k_-) \to$$

$$\to \tilde{H}_q(S^n \setminus S^k) \to \tilde{H}_q(S^n \setminus D^k_+) \oplus \tilde{H}_q(S^n \setminus D^k_-) \to \cdots$$
The groups at the ends are equal zero by Lemma A, thus
\[ \tilde{H}_q(S^n \setminus S^k) \cong \tilde{H}_{q+1}(S^n \setminus S^{k-1}) \]
since \( D_+^k \cap D_-^k = S^{k-1} \). This completes the induction. \( \square \)

9. Let \( X \) be a finite simply-connected CW-complex with \( \tilde{H}_n(X; \mathbb{Z}) = 0 \) for all \( n \). Prove that \( X \) is contractible.

**Solution:** By assumption, \( \pi_1(X, x_0) = 0 \). Then the Hurewicz homomorphism \( h : \pi_2(X, x_0) \to H_2(X; \mathbb{Z}) \) is an isomorphism. Thus \( \pi_2(X, x_0) = 0 \). Then the Hurewicz homomorphism \( h : \pi_3(X, x_0) \to H_3(X; \mathbb{Z}) \) is an isomorphism, and so on. We obtain that \( \pi_q(X, x_0) = 0 \) for all \( q = 1, 2, \ldots \). Here we use Hurewicz Theorem:

**Theorem.** (Hurewicz) Let \( (X, x_0) \) be a based space, such that
\[ \pi_0(X, x_0) = 0, \; \pi_1(X, x_0) = 0, \cdots, \pi_{n-1}(X, x_0) = 0, \]where \( n \geq 2 \). Then
\[ H_1(X) = 0, \; H_2(X) = 0, \cdots, H_{n-1}(X) = 0, \]
and the Hurewicz homomorphism \( h : \pi_n(X, x_0) \to H_n(X) \) is an isomorphism.

Thus the constant map \( c : X \to * \) induces isomorphism in homotopy groups, and Whitehead Theorem implies that \( X \) is homotopy equivalent to a point. \( \square \)

10. Let \( X \) be a path-connected space. Prove that the group \( H^1(X; \mathbb{Z}) \) is free abelian group.

**Solution:** Since \( X \) is path-connected, \( H_0(X; \mathbb{Z}) \cong \mathbb{Z} \), and \( H^0(X; \mathbb{Z}) \cong \mathbb{Z} \). Let \( G \) be an abelian group. Then universal coefficient formula:
\[ 0 \to \text{Ext}(H_{q-1}(X), G) \to H^q(X; G) \to \text{Hom}(H_q(X), G) \to 0 \]for each \( q \geq 0 \). If \( q = 1 \) and \( G = \mathbb{Z} \), then \( \text{Ext}(H_0(X), \mathbb{Z}) = \text{Ext}(\mathbb{Z}, \mathbb{Z}) = 0 \), and the group
\[ \text{Hom}(H_1(X), \mathbb{Z}) \]
does not have any torsion. Thus the group \( H^1(X; \mathbb{Z}) \cong \text{Hom}(H_1(X), \mathbb{Z}) \) has no torsion. \( \square \)