1. Let \( f : X \rightarrow [0, \infty] \) be a measurable function on a measure space \((X, \mu)\) with \( \mu(X) < \infty \). Prove that \( f \in L^1(X) \) if and only if
\[
\sum_{n \geq 0} 2^n \mu\{x \in X | f(x) \geq 2^n\} < \infty.
\]

**Solution:**
Let \( F_n = \{ x \in X | f(x) \geq 2^k \} \). Consider disjoint sets
\[
E_k = \{ x \in X | 2^{k-1} \leq f(x) < 2^k \} \quad \text{if} \quad k \geq 1 \quad \text{and} \quad E_0 = \{ x \in X | f(x) < 1 \}.
\]
Then, since
\[
F_n = \coprod_{k \geq n} E_k \quad \text{and} \quad X = \coprod_{k \geq 0} E_k,
\]
we have
\[
\sum_{k \geq 0} 2^{k-1} \chi_{E_k} \leq f \leq \sum_{k \geq 0} 2^k \chi_{E_k}
\]
or \( g/2 \leq f \leq g \) where \( g = \sum_{k \geq 0} 2^k \chi_{E_k} \). The sets \( E_k \) are measurable because \( f \) is measurable, hence \( g \) is measurable. Since both \( f \) and \( g \) are non-negative, \( f \) is in \( L^1 \) if and only if \( g \) is. On the other hand,
\[
\sum_{n \geq 0} 2^n \mu\{x \in X | f(x) \geq 2^n\} = \sum_{n \geq 0} 2^n \mu(F_n) = \sum_{n \geq 0} 2^n \sum_{k \geq n+1} \mu(E_k) = \sum_{k \geq 1} \mu(E_k) \sum_{n=0}^{k-1} 2^n = \sum_{k \geq 1} (2^k - 1) \mu(E_k) = \int_X g \mu - \mu(E_0) - \mu(X).
\]

2. Find each limit and justify your answers. Quote all the theorems you are using to manipulate the limits and integrals and verify their applicability.

(a) \( \lim_{n \to \infty} \int_{\frac{1}{n}}^{\infty} e^{\frac{1}{n} \sqrt{x}} \pi dx \).

(b) \( \int_0^{\infty} \int_{-\infty}^{\infty} e^{-x^2} dx dy \).

**Solution:**
(a) First, notice that the sequence of functions \( f_n(x) = e^{\frac{1}{n} \sqrt{x}} \chi_{[\frac{1}{n}, \infty]} \) on \([0, \infty)\) converges pointwise to \( f(x) = \lim_{n \to \infty} f_n(x) = e^{-x} \). Since each \( f_n \) is piece-wise continuous, hence measurable, and \( 0 \leq f_n(x) \leq f(x) \) for all \( x \in [0, \infty) \) and \( f \in L^1 \), the Dominated Convergence Theorem applies, giving
\[
\lim_{n \to \infty} \int_{\frac{1}{n}}^{\infty} e^{\frac{1}{n} \sqrt{x}} dx = \lim_{n \to \infty} \int_0^{\infty} f_n(x) dx = \int_0^{\infty} f(x) dx = \int_0^{\infty} e^{-x} dx = 1.
\]
(b) The function \( f(x, y) = e^{-x^2} \) is continuous on \( \mathbb{R}^2 \), hence measurable. Further let

\[ E = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x\}. \]

Then \( E \) is a closed subset of \( \mathbb{R}^2 \), hence measurable. It follows that \( g = \chi_E f \) is measurable on \( \mathbb{R}^2 \). The integral to be evaluated can be written as

\[ \int_{\mathbb{R}} \int_{\mathbb{R}} g(x, y) \, dx \, dy. \]

Since \( g \) is nonnegative and Lebesgue measure on \( \mathbb{R} \) is \( \sigma \)-finite, we can apply the version of Fubini’s Theorem for positive measurable functions:

\[ \int_{\mathbb{R}} \int_{\mathbb{R}} g(x, y) \, dx \, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x, y) \, dy \, dx = \int_{0}^{\infty} \int_{0}^{x} \exp(-x^2) \, dy \, dx = \int_{0}^{\infty} x \exp(-x^2) \, dx = \frac{1}{2} \]

(the last integral is evaluated using an obvious substitution).

3. (a) State the Radon-Nikodym theorem.

(b) Let \( (X, \mathcal{M}, \mu) \) be a measure space. Denote by \( M(X) \) the space of all complex measures defined on the \( \sigma \)-algebra \( \mathcal{M} \).

Let \( E \subset M(X) \) be the set of all measures in \( M(X) \) which are absolutely continuous with respect to \( \mu \). Prove that \( E \) is a closed subspace of \( M(X) \).

Solution:

(a) Let \( \mu \) be a positive \( \sigma \)-finite measure on a \( \sigma \)-algebra \( \mathcal{M} \) on a set \( X \), and let \( \nu \) be a complex measure on \( X \). Then there exists a unique pair of complex measures \( \nu_a \) and \( \nu_s \) on \( \mathcal{M} \) such that

\[ \nu = \nu_a + \nu_s, \quad \nu_a \ll \mu, \quad \nu_s \perp \mu. \]

(b) The proof that \( E \) is a vector subspace is easy, and is omitted. Let us show that \( E \) is closed. Let \( (\nu_n) \) be a sequence in \( E \) and suppose \( \|\nu_n - \nu\| \to 0 \). For \( B \in \mathcal{M} \), we have

\[ |\nu_n(B) - \nu(B)| \leq |\nu_n(B) - \nu(B)| + |\nu_n(X \setminus B) - \nu(X \setminus B)| \leq \|\nu_n - \nu\|. \]

Therefore \( \nu_n(B) \to \nu(B) \) for all \( B \in \mathcal{M} \).

Let \( N \in \mathcal{M} \) satisfy \( \mu(N) = 0 \). Then

\[ \nu(N) = \lim_{n \to \infty} \nu_n(N) = \lim_{n \to \infty} 0 = 0. \]

This shows that \( \nu \) is absolutely continuous with respect to \( \mu \), as desired.

4. Let \( p, q, r \) be real numbers in \( [1, \infty) \) satisfying \( 1/r = 1/p + 1/q \). Prove that for functions \( f \in L^p \) and \( g \in L^q \) their product \( fg \) belongs to \( L^r \) and that the multiplication operator

\[ M_f : L^q \to L^r, \quad M_f(g) = fg, \]

is a bounded operator with norm \( \|M_f\| = \|f\|_p \).
Solution:

Since \( \frac{1}{p} + \frac{1}{q} = 1 \) we have by Hölder’s inequality

\[
\|fg\|_r = \left( \int |fg|^r \right)^{1/r} = \left( \int |f|^r |g|^r \right)^{1/r} \leq \left( \|f\|_p |g|_q \right)^{1/r} = \|f\|_p |g|_q.
\]

This implies that \( fg \in L^r \). Also we have \( \|M_f(g)\|_r \leq \|f\|_p |g|_q \) which means that \( M_f \) is bounded and \( \|M_f\| \leq \|f\|_p \).

To prove the lower bound for \( \|M_f\| \), notice that \( h = f^{p/q} \) is in \( L^q \) with

\[
\|h\|_q = \left( \int |f^{p/q}|^q \right)^{1/q} = \left( \int |f|^p \right)^{1/q} = \|f\|_p^{p/q}
\]

and

\[
\|M_f(h)\|_r = \|f^{1+p/q}\|_r = \left( \int |f|^{r(1+p/q)} \right)^{1/r} = \left( \int |f|^p \right)^{1/r} = \|f\|_p^{p/r}.
\]

This gives the desired lower bound for \( \|M_f\| \):

\[
\|M_f\| \geq \|M_f(h)\|_r / \|h\|_q = \|f\|_p^{p/r-p/q} = \|f\|_p,
\]

since \( p(\frac{1}{p} - \frac{1}{q}) = 1 \). \( \blacksquare \)

5. Let \( V \) be a Banach space, and let \( \{\phi_n\} \) be a sequence of continuous linear functionals on \( V \), such that the limit

\[
\phi(x) = \lim_{n \to \infty} \phi_n(x)
\]

exists for all \( x \in V \).

Prove that \( \phi \) is a continuous linear functional on \( V \).

Solution:

Linearity of \( \phi \) is clear:

\[
\phi(ax + by) = \lim_{n \to \infty} \phi_n(ax + by) = \lim_{n \to \infty} (a\phi_n(x) + b\phi_n(y)) = a\phi(x) + b\phi(y).
\]

For linear functionals on a normed space continuity is equivalent to boundedness, therefore \( \phi_n \) is bounded.

To prove that \( \phi \) is bounded we use the Banach-Steinhaus Theorem.

Let \( x \in V \). Since \( \phi_n(\xi) \to \phi(\xi) \), the set \( \{\phi_n(x)|n \in N\} \) is bounded. Since \( V \) is complete and the \( \phi_n \) are bounded, the Banach-Steinhaus Theorem implies that \( M = \sup_{n \in N} \|\phi_n\| < \infty \).

Therefore, for every \( x \in V \),

\[
|\phi(x)| = \lim_{n \to \infty} |\phi_n(x)| \leq M\|x\|.
\]

This shows that \( \|\phi\| \leq M \), so that \( \phi \) is bounded. \( \blacksquare \)

6. (a) Give the definition of a Hilbert space.
(b) Let $F$ and $H$ be Hilbert spaces, with scalar products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively. Define a scalar product on the vector space direct sum $F \oplus H$ by
\[
\langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle = \langle \xi_1, \xi_2 \rangle_1 + \langle \eta_1, \eta_2 \rangle_2.
\]
Prove that this makes $F \oplus H$ a Hilbert space. (Be sure to prove completeness!)

**Solution:** (a) A Hilbert space is a complex vector space $H$ equipped with a scalar product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ satisfying the following properties:

1. $\langle \eta, \xi \rangle = \overline{\langle \xi, \eta \rangle}$ for all $\xi, \eta \in H$.
2. $\langle \xi_1 + \xi_2, \eta \rangle = \langle \xi_1, \eta \rangle + \langle \xi_2, \eta \rangle$ for all $\xi_1, \xi_2, \eta \in H$.
3. $\langle \alpha \xi, \eta \rangle = \alpha \langle \xi, \eta \rangle$ for all $\xi, \eta \in H$ and $\alpha \in \mathbb{C}$.
4. $\langle \xi, \xi \rangle \geq 0$ for all $\xi \in H$.
5. $\langle \xi, \xi \rangle = 0$ only if $\xi = 0$.
6. $H$ is complete in the norm defined by $\|\xi\| = (\langle \xi, \xi \rangle)^{1/2}$.

(b) The verification of the axioms (1) through (4) above for the scalar product on $F \oplus H$ is completely routine, and is omitted from the solutions. For (5), assume that
\[
\langle (\xi, \eta), (\xi, \eta) \rangle = 0.
\]
This means that
\[
\langle \xi, \xi \rangle_1 + \langle \eta, \eta \rangle_2 = 0.
\]
Since
\[
\langle \xi, \xi \rangle_1 \geq 0 \quad \text{and} \quad \langle \eta, \eta \rangle_2 \geq 0,
\]
this implies that
\[
\langle \xi, \xi \rangle_1 = \langle \eta, \eta \rangle_2 = 0.
\]
So $(\xi, \eta) = 0$.

It remains to prove (6) (completeness).

Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots$ be a Cauchy sequence in $F \oplus H$. We observe the inequality, for any $\xi \in F$ and $\eta \in H$,
\[
\|(\xi, \eta)\| = \left(\langle \xi, \xi \rangle_1 + \langle \eta, \eta \rangle_2\right)^{1/2} \geq \max(\|\xi\|, \|\eta\|).
\]
It follows from this inequality that the sequences $\xi_1, \xi_2, \ldots$ and $\eta_1, \eta_2, \ldots$ are Cauchy in $F$ and $H$ respectively. Therefore they have limits $\xi \in F$ and $\eta \in H$. We prove that $\|(\xi, \eta) - (\xi_n, \eta_n)\| \to 0$.

Using the formula for the norm and the scalar product, we find
\[
\|(\xi, \eta) - (\xi_n, \eta_n)\|^2 = \|\langle \xi - \xi_n, \eta - \eta_n \rangle\|^2 = \langle \xi - \xi_n, \xi - \xi_n \rangle_1 + \langle \eta - \eta_n, \eta - \eta_n \rangle_2
\]
\[
= \|\xi - \xi_n\|^2 + \|\eta - \eta_n\|^2.
\]
Since
\[
\|\xi - \xi_n\| \to 0 \quad \text{and} \quad \|\eta - \eta_n\| \to 0,
\]
it follows that $\|(\xi, \eta) - (\xi_n, \eta_n)\| \to 0$. \qed
7. Describe all entire functions \( f : \mathbb{C} \mapsto \mathbb{C} \) for which there exist constants \( a \) and \( b \) such that
\[
|f(z)| \leq a|z|^{5/2} + b \quad \forall z \in \mathbb{C}.
\]

**Solution:**

By Cauchy's integral formula for derivatives,
\[
f^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}} dz, \quad R > 0,
\]
we get the estimate
\[
|f^{(k)}(0)| \leq \frac{k!}{2\pi} \int_{|z|=R} \left| \frac{f(z)}{|z|^{k+1}} \right| |dz| \leq \frac{k!}{2\pi R^{k+1}} \int_{|z|=R} (a|z|^{5/2} + b)|dz| = \frac{k!aR^{5/2} + b}{R^k}.
\]

If \( k \geq 3 \) then the right hand side of the above estimate goes to zero. By Taylor expansion, \( f(z) = f(0) + f'(0)z + f''(0)z^2/2 \). So that \( f \) is a polynomial of degree at most 2.

8. Evaluate the integral
\[
\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1} dx.
\]

**Solution:**

Consider the function \( f(z) = \frac{e^{iz}}{(z^4 + 1)} \) and integral of \( f \) over the contour \( \Gamma_R \), \( R > 0 \), which consists of 2 parts: the segments of real line from \(-R\) to \( R \) and the upper half circle \( \Gamma_R^+(t) = Re^{it}, \, 0 \leq t \leq \pi \). We have
\[
\left| \int_{\Gamma_R^+} \frac{e^{iz}}{z^4 + 1} dz \right| \leq \int_{\Gamma_R^+} \frac{|e^{iz}|}{R^4 + 1} |dz| = \frac{\pi R}{R^4 + 1}
\]
which goes to zero as \( R \to \infty \). Therefore,
\[
\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1} dx = \lim_{R \to \infty} \Re \left( \int_{\Gamma_R} \frac{e^{iz}}{z^4 + 1} dz \right)
= \Re \left( 2\pi i \text{Res} \left( \frac{e^{iz}}{z^4 + 1}, e^{\pi i/4} \right) + 2\pi i \text{Res} \left( \frac{e^{iz}}{z^4 + 1}, e^{3\pi i/4} \right) \right)
= -2\pi i \left( \frac{e^{i\pi/4}}{4\pi^3} + \frac{e^{3i\pi/4}}{4\pi^3} \right)
= \frac{\pi}{\sqrt{2}} e^{-1/\sqrt{2}} \left( \cos \frac{\sqrt{2}}{2} + \sin \frac{\sqrt{2}}{2} \right)
\]
9. Let \( f \) be a continuous complex valued function on the interval \([0,1]\), and define function \( F \) by

\[
F(z) = \int_0^1 f(t)e^{itz} \, dt, \quad z \in \mathbb{C}.
\]

Prove that \( F \) is analytic in the entire complex plane.

\textit{Solution:}

\textbf{Solution 1} The function \( f(t)e^{itz} \) is a continuous function of the variable \((t,z) \in [0,1] \times \mathbb{C}\). If \( R \subset \mathbb{C} \) is a rectangle, then by Fubini's theorem and the analyticity of \( e^{itz} \) with respect to \( z \),

\[
\int_R F(z) \, dz = \int_R \int_0^1 f(t)e^{itz} \, dt \, dz = \int_0^1 \int_R f(t)e^{itz} \, dz \, dt = 0.
\]

By Morera's theorem, \( F \) is analytic.

\textbf{Solution 2} Let \( z \in \mathbb{C} \). Since \( f(t) \) is bounded, the series

\[
f(t)e^{itz} = f(t) \sum_{n=0}^{\infty} \frac{t^n z^n}{n!}
\]

converges uniformly for \( t \in [0,1] \), so that we can change the order of summation and integration to get

\[
F(z) = \int_0^1 f(t)e^{itz} \, dt = \sum_{n=0}^{\infty} \int_0^1 f(t) t^n dt \frac{z^n}{n!}.
\]

Since

\[
\left| \frac{1}{n!} \int_0^1 f(t) t^n dt \right| \leq \frac{1}{n!} \int_0^1 |f(t)| dt,
\]

the radius of convergence of the series of \( F \) is \( \infty \).