Algebra Qualifying Exam

Fall 2004

Conventions: Throughout this examination, assume all rings have identities and all modules are unitary.

**Part I** Theorems. Carefully state each of the following;

1. The Sylow theorems.
2. The Hilbert Nullstellensatz (in any form).
3. At least three equivalent definitions of projective modules.

**Part II** True–False. Determine whether each statement is true or false. Support your decision by a brief explanation or a counterexample.

1. There are no simple groups of order 80.
2. If $R$ is a PID then the polynomial ring $R[x]$ is also a PID.
3. If $R$ is a commutative ring without zero divisors then any submodule of a free $R$-module is projective.
4. Let $\mathbb{Z}_2$ be a $\mathbb{Z}_6$-module via the projection $\mathbb{Z}_6 \to \mathbb{Z}_6/2\mathbb{Z}_6 = \mathbb{Z}_2$. Then $\text{Ext}_{\mathbb{Z}_2}(\mathbb{Z}_2, \mathbb{Z}_6) = 0$.
5. If $D$ is a division ring and $M$ is a nonzero left $D$-module then $\text{End}(M)$ is a simple ring.
6. If $R$ is a PID and $a \in R \setminus \{0\}$ then the ring $R/(a)$ is Noetherian and Artinian.
7. If $u$ and $v$ are algebraic over a field $K$ then so is $u + v$.

**Part III** Problems. Give complete solutions for each of the following.

1. Determine the Galois group of $X^3 - 5$ over $\mathbb{Q}$, describe explicitly all the subfields of its splitting field, and decide which of these subfields are normal over $\mathbb{Q}$.
2. If $N$ is a non-trivial normal subgroup of a finite nilpotent group $G$ then $N \cap C(G) \neq \{e\}$ where $C(G)$ is the center of $G$.
3. If $M$ is a Noetherian (left) module over a ring then every epimorphism $f : M \to M$ is an isomorphism.
4. $\mathbb{C}^7$ is not a finite union of its proper subvarieties.
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Part I Theorems. Carefully state each of the following;

1. The Sylow theorems.
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3. At least three equivalent definitions of projective modules.

Part II True–False. Determine whether each statement is true or false. Support your decision by a brief explanation or a counterexample.

1. There are no simple groups of order 80.
   True. Let \( G \) be a group of order 80. If its 5-Sylow subgroup is unique it is normal. If not then the Sylow theorems give 16 5-Sylow subgroups that cover 64 elements of order 5. This leaves room for just one 2-Sylow subgroup which is normal.

2. If \( R \) is a PID then the polynomial ring \( R[x] \) is also a PID.
   False. If \( R = \mathbb{R}[y] \) then \( R[x] = \mathbb{R}[y,x] \) is not PID.

3. If \( R \) is a commutative ring without zero divisors then any submodule of a free \( R \)-module is projective.
   False. Take \( R = \mathbb{R}[x,y] \) as a module over itself and its submodule \( M = (x,y) \). It is not projective since the natural projection \( R \oplus R \to M \) does not split.

4. Let \( \mathbb{Z}_2 \) be a \( \mathbb{Z}_6 \)-module via the projection \( \mathbb{Z}_6 \to \mathbb{Z}_6/2\mathbb{Z}_6 = \mathbb{Z}_2 \). Then \( \text{Ext}_{\mathbb{Z}_6}(\mathbb{Z}_2, \mathbb{Z}_6) = 0. \)
   True. \( \mathbb{Z}_2 \) is a direct summand of the free module \( \mathbb{Z}_6 \) whence projective and every extension of it splits.

5. If \( D \) is a division ring and \( M \) is a nonzero left \( D \)-module then \( \text{End}(M) \) is a simple ring.
   False. If \( M \) is not finite dimensional then \( \text{End}(M) \) has the proper ideal consisting of all the endomorphisms of finite ranks.

6. If \( R \) is a PID and \( a \in R \setminus \{0\} \) then the ring \( R/(a) \) is Noetherian and Artinian.
   True. The ideals of \( R/(a) \) correspond to the ideals of \( R \) containing \( a \) whence generated by factors of \( a \). Hence they form a finite set.

7. If \( u \) and \( v \) are algebraic over a field \( K \) then so is \( u + v. \)
   True. The fields \( K(u) \) and \( K(v) \) are finite-dimensional over \( k \), whence so is \( K(u)(v) \). The latter field contains \( u + v \) implying \( u + v \) is algebraic over \( K \).

Part III Problems. Give complete solutions for each of the following.

1. Determine the Galois group of \( X^3 - 5 \) over \( \mathbb{Q} \), describe explicitly all the subfields of its splitting field, and decide which of these subfields are normal over \( \mathbb{Q} \).
   The discriminant is \( D = -27(-5)^2 = -3^3 \cdot 25 \) which is not a square in \( \mathbb{Q} \). Thus the Galois group is \( S_3 \). The roots of the polynomial are \( \alpha, \alpha \omega, \alpha \bar{\omega} \) where \( \alpha = \sqrt[3]{5} \in \mathbb{R} \) and \( \omega = 1/2(-1 + i\sqrt{3}) \). Thus the splitting field is \( \mathbb{Q}(i\sqrt{3}, \sqrt[3]{5}) \).
The subfield corresponding to $A_3 \subset S_3$ is $\mathbb{Q}(i\sqrt{3}) = \mathbb{Q}(i\sqrt{3})$ and this is the only normal extension. There are 3 other extensions corresponding to subgroups generated by transpositions permuting two of the three roots. The first extension is $\mathbb{Q}(\alpha)$ (the respective transposition is the complex conjugation), the second one is $\mathbb{Q}(\alpha \bar{\omega})$, and the third one is $\mathbb{Q}(\alpha \bar{\omega})$.

2. If $N$ is a non-trivial normal subgroup of a finite nilpotent group $G$ then $N \cap C(G) \neq \{e\}$ where $C(G)$ is the center of $G$.

Since $N$ is nilpotent it is the direct sum of its Sylow subgroups as well as $G$. Let $p$ be a prime dividing the order of $N$, $N'$ the $p$-Sylow subgroup of $N$, and $G'$ the $p$-Sylow subgroup of $G$. Clearly $N'$ is a normal subgroup of $G'$.

Now consider the action of $G'$ on $N'$ by conjugation. Every orbit has cardinality either 1 or divisible by $p$ and the disjoint union of the orbits is $N'$ whence its cardinality is divisible by $p$. Since the cardinality of the orbit of $e$ is 1 there is another element $n \in N'$ whose orbit has also the cardinality 1. This implies that $n \in C(G')$ and since $G'$ is a direct summand of $G$ we have $n \in N \cap C(G)$.

3. If $M$ is a Noetherian (left) module over a ring then every epimorphism $f : M \to M$ is an isomorphism.

Suppose $\text{Ker} f \neq 0$. Put $M_n = \text{Ker} f^n$ for every natural $n$. These submodules of $M$ form a nondecreasing sequence. Moreover it is easy to see by induction that the sequence is really increasing. Indeed the above supposition is the base of the induction and if $x \in M_{n-1} \setminus M_{n-2}$ then $f^{-1}(x) \subset M_n \setminus M_{n-1}$. This contradicts the condition that $M$ is Noetherian.

4. $\mathbb{C}^7$ is not a finite union of its proper subvarieties.

The ideal $J$ of the subvariety $\mathbb{C}^7$ of $\mathbb{C}^7$ is 0. Thus $J$ is prime (since the polynomial ring over $\mathbb{C}$ is an integral domain). This is equivalent to the irreducibility of the variety.