Analysis Qualifying Exam, Fall 2022
Answer Key

FQ1. Determine, with justification, the limit
\[
\lim_{n \to \infty} \frac{1}{n} \int_0^{n^2} \frac{\cos(x/n^2)}{\sqrt{x + \sin(x/n^2)}} \, dx.
\]
(The answer is OK to be a definite integral.)

Solution. By change of variable \( x = n^2 y \) we compute
\[
\frac{1}{n} \int_0^{n^2} \frac{\cos(x/n^2)}{\sqrt{x + \sin(x/n^2)}} \, dx = \int_0^1 \frac{\cos y}{\sqrt{y + \sin y n^2}} \, dy
\]
Note that for \( y \in [0, 1] \) function \( \frac{1}{\sqrt{y}} \) dominates \( \frac{\cos y}{\sqrt{y + \sin y n^2}} \) and \( \int_0^1 \frac{1}{\sqrt{y}} \, dy = 2 \), we may apply Lebesque’s dominated convergence theorem to get
\[
\lim_{n \to \infty} \frac{1}{n} \int_0^{n^2} \frac{\cos(x/n^2)}{\sqrt{x + \sin(x/n^2)}} \, dx = \int_0^1 \lim_{n \to \infty} \frac{\cos y}{\sqrt{y + \sin y n^2}} \, dy.
\]
Hence the limit is \( \int_0^1 \frac{\cos y}{\sqrt{y}} \, dy \). \qed

FQ2. Let \( \{f_n\} \in L^1([0,1], dm) \) be a sequence of functions where \( dm \) is the Lebesque measure. Suppose there is a non-negative function \( h \in L^1([0,1], dm) \) such that \( \{f_n\} \) satisfies
(i) \( \int_{[0,1]} f_n g \, dm \to 0 \) for each \( g \in C([0,1]) \), and
(ii) \( |f_n| \leq h \) for all \( n \).

Show that \( \int_A f_n \, dm \to 0 \) as \( n \to \infty \) for each Borel set \( A \subset [0,1] \).

Solution. For any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for any \( E \subset [0,1] \) with \( m(E) \leq \delta \) we have \( \int_E h \, dm \leq \epsilon \).

For the \( \delta \) and Borel set \( A \) there is a compact set \( K \) and open set \( U \subset [0,1] \) such that \( K \subset A \subset U \) and \( m(U \setminus K) < \delta \).

There is a continuous function \( g : [0,1] \to \mathbb{R} \) such that \( 0 \leq g \leq 1 \), \( g = 1 \) on \( K \), and \( g = 0 \) outside \( U \).
Hence we can compute
\[
\lim_{n \to \infty} \int_A f_n dm = \lim_{n \to \infty} \int_{[0,1]} f_n \chi_A dm \\
\leq \lim_{n \to \infty} \left| \int_{[0,1]} f_n g dm \right| + \lim_{n \to \infty} \int_{[0,1]} f_n (\chi_A - g) dm \\
\leq 0 + \lim_{n \to \infty} \int_{[0,1]} h \chi_{U \setminus K} dm \leq \epsilon.
\]
Hence the limit is zero.

**FQ3.** Let \((X, \mu)\) be a positive measure space with finite total measure. Let \(\{f_n\}_{n=1}^{\infty}\) be a sequence of measurable functions on \(X\) which satisfy \(\int_X |f_n(x)|^{2022} d\mu(x) \leq 2023\). Suppose \(f_n \to 0\) a.e. on \(X\). Prove that \(\int_X |f_n(x)| d\mu(x) \to 0\).

Solution. By Egorov theorem for any \(\epsilon > 0\) there is a \(\delta > 0\) such that there is a subset \(E \subset X\) with \(\mu(E) < \epsilon\) which satisfies \(f_n \to 0\) uniformly on \(X \setminus E\). We compute using Holder inequality
\[
\int_E |f_n(x)| d\mu(x) \leq \left( \int_E |f_n(x)|^{2022} d\mu(x) \right)^{\frac{1}{2022}} \cdot \left( \int_E 1^{2022} d\mu(x) \right)^{\frac{2021}{2022}} \\
\leq 2023 \cdot \epsilon^{\frac{2021}{2022}}.
\]

Since \(f_n \to 0\) uniformly on \(X \setminus E\) and \(\mu(X) < \infty\), we can find \(N\) such that for any \(n > N\)
\[
\int_{X \setminus E} |f_n(x)| d\mu(x) \leq \epsilon^{\frac{2021}{2022}}.
\]
Hence for \(n > N\) we have
\[
\int_X |f_n(x)| d\mu(x) \leq (2023 \cdot \epsilon^{\frac{2021}{2022}} + 1) \epsilon^{\frac{2021}{2022}}.
\]
The \(L^1\) convergence of \(f_n\) follows.

**FQ4.** Let \(K\) be a continuous function on \([0, 1] \times [0, 1]\). We define operator \(T : L^2([0, 1]) \to C([0, 1])\) by
\[
Tf(x) = \int_{[0,1]} K(x, y) f(y) dy.
\]

(F4a) Show that \(Tf\) is indeed continuous on \([0, 1]\) and \(\|Tf\|_{\infty} \leq \|K\|_{\infty} \cdot \|f\|_{L^2} \).

Here \(\| \cdot \|_{\infty}\) denotes the \(L^\infty\)-norm.

(F4b) Show that if \(\{f_n\}\) is a bounded sequence in \(L^2([0, 1])\), the sequence \(\{Tf_n\}\) has a uniformly convergent subsequence.
(F4c) Assume that $T$ is one-to-one, show that $T$ does not map $L^2([0,1])$ onto $C([0,1])$.

Solution. (F4a) Since $K$ is continuous on compact set $[0, 1] \times [0, 1]$, for any $\epsilon > 0$ there is a $\delta > 0$ such that $|K(x_2, y_2) - K(x_1, y_1)| \leq \epsilon$ when $\|(x_2, y_2) - (x_1, y_1)\| < \delta$. We compute using Cauchy-Schwarz inequality, for $|x_2 - x_1| < \delta$

$$|Tf(x_2) - Tf(x_1)| \leq \int_{[0,1]} |K(x_2, y) - K(x_1, y)| \cdot |f(y)| dy \leq \epsilon \int_{[0,1]} |f(y)| dy \leq \|f\|_{L^2} \cdot \epsilon.$$  

Hence $Tf$ is a continuous function.

We compute using Cauchy-Schwarz inequality

$$\|Tf\|_\infty \leq \|K\|_\infty \int_{[0,1]} |f(y)| dy \leq \|K\|_\infty \cdot \|f\|_{L^2}.$$  

(F4b) From the first part of the proof in (F4a) we have, for $|x_2 - x_1| < \delta$

$$|Tf_n(x_2) - Tf_n(x_1)| \leq \|f_n\|_{L^2} \cdot \epsilon \leq M \cdot \epsilon,$$

where $M = \sup\{|f_n|_{L^2}\}$, hence sequence $\{Tf_n\}$ are equi-continuous. From the second part of the proof in (F4a) we have $\|Tf_n\|_{\infty} \leq \|K\|_{\infty} \cdot M$, hence sequence $\{Tf_n\}$ are uniformly bounded. The conclusion in (F4b) follows from Arzela–Ascoli theorem.

(F4c) By Banach open mapping theorem the inverse map $T^{-1} : C([0,1]) \to L^2([0,1])$ is a bounded linear operator. Hence there is a constant $C > 0$ such that, for each $f \in L^2([0,1])$ we have using the last inequality in the proof of (F4a)

$$\|f\|_{L^2} \leq C\|Tf\|_{\infty} \leq C\|K\|_{\infty} \cdot \|f\|_{L^1}.$$  

This is impossible. \qed

**FQ5.** Assume that $X, Y, Z$ are Banach spaces. Assume $U : X \times Y \to Z$ is bilinear, namely for every $x \in X$, $U(x, \cdot) : Y \to Z$ is linear and for every $y \in Y$, $U(\cdot, y) : X \to Z$ is linear. Show that if $\ell(U(x, \cdot)) \in Y^*$ and $\ell(U(\cdot, y)) \in X^*$ are bounded linear functional for every bounded linear functional $\ell \in Z^*$, $x \in X$, and $y \in Y$, then there exists constant $M > 0$ such that

$$\|U(x, y)\| \leq M\|x\| \cdot \|y\| \quad \text{for } x \in X, y \in Y.$$  

Solution. Step 1. For each $x \in X$ $U(x, \cdot) : Y \to Z$ is a bounded linear operator, and for each $y \in Y$ $U(\cdot, y) : X \to Z$ is a bounded linear operator. To see the first statement, we consider a family of bounded linear functional $\{U(x, y) \in (Z^*)^*, \|y\| \leq 1\}$. By the assumption that $\ell(U(x, \cdot)) \in Y^*$ we know that $\sup_{\|y\| \leq 1} |\ell(U(x, y))| < \infty$ for each $\ell$ and $x$, hence by Banach-Steinhaus
theorem \sup_{\|y\| \leq 1} \|U(x, y)\| < \infty \text{ for each } x. \text{ The second statement follows by a similar proof.}

Step 2. Now we consider a family of bounded linear operator \{U(x, \cdot) : Y \to Z, \|x\| \leq 1\}. Since \sup_{\|x\| \leq 1} \|U(x, y)\| < \infty \text{ for each fixed } y \text{ by step 1, again by Banach-Steinhaus theorem we have } \sup_{\|x\| \leq 1} \|U(x, \cdot)\| \leq M \text{ for some constant } M < \infty. \text{ This implies } \sup_{\|x\| \leq 1, \|y\| \leq 1} \|U(x, y)\| \leq M, \text{ and the required inequality follows.} \quad \Box

\textbf{FQ6.} Let } X \text{ be a normed space. Show that for any bounded sequence } \{x_n\} \text{ with } \|x_n\| \leq M < \infty, \text{ there exists a subsequence } \{x_{n_k}\} \text{ and } x_\infty \text{ such that}
\begin{enumerate}[label=(F6i), itemsep=0pt]
\item \(x_{n_k} \text{ converges to } x_\infty \text{ weakly, and}\)
\item \(\|x_\infty\| \leq M.\)
\end{enumerate}

\textbf{Solution.} Since \{x_n\} \subset (X^*)^* \text{ are bounded, by Banach-Alaoglu theorem we get weak* convergence of some subsequence, i.e., there is a subsequence } x_{n_k} \text{ and } x_\infty \text{ such that } \ell(x_{n_k}) \to \ell(x_\infty) \text{ for each } \ell \in X^*. \text{ We have for any } \ell \in X^* \text{ with } ||\ell|| \leq 1 \text{ that } ||\ell(x_\infty)|| = \lim ||\ell(x_{n_k})|| \leq \lim ||\ell|| \cdot \|x_{n_k}\| \leq M, \text{ hence } ||x_\infty|| \leq M. \quad \Box

\textbf{FQ7.} Consider the following made-up transform.

\[ f^\#(\omega) = \int_{-\infty}^{\infty} \cos(e^{i\omega}t) f(t) dt \]

Show that if \(f \in L^1(\mathbb{R})\) (for simplicity you may assume \(f\) is \(\mathbb{R}\)-valued), then \(f^\#\) is continuous and vanishes at \(+\infty\). You are encouraged to use well-known results without proving them from scratch.

\textbf{Solution:} We know that

\[ \hat{f}(e^{i\omega}) = \int_{-\infty}^{\infty} \exp(ie^{i\omega}t) f(t) dt = \int_{-\infty}^{\infty} \cos(e^{i\omega}t) f(t) dt + i \int_{-\infty}^{\infty} \sin(e^{i\omega}t) f(t) dt \]

so

\[ f^\#(\omega) = \text{Re} \left[ \hat{f}(e^{i\omega}) \right]. \]

This is a composition of continuous functions: \(\hat{f}\) is continuous for \(f \in L^1\); This is a ”well-known” result. Also \(\omega \to e^{i\omega}, z \to \text{Re}(z)\) are both continuous, thus \(f^\#(\omega)\) is continuous. It vanishes at \(+\infty\) because \(e^{i\omega} \to \infty\) when \(\omega \to +\infty\), and \(\hat{f}\) vanishes at \(\infty\). \quad \Box

\textbf{FQ8.} Suppose that \(f\) is holomorphic and bounded by \(|f(z)| \leq 10 + |z|^{2022.9}\). Show that \(f(z)\) is a polynomial.
Solution: Use the Cauchy integral formula, to compute the 2023rd derivative at any point: choose a ball of radius $R$ around a point $a$ and we get

$$|f^{2023}(a)| \leq \frac{2023!}{R^{2023}} |a + R|^{2022.9} + 10.$$ 

Fixing $a$ and letting $R \to \infty$, we get $f^{2023}(z) \equiv 0$. Now pick any point, say 0, and expand a Taylor series, we get a finite series, i.e. a polynomial. □

**FQ9**. Compute

$$\int_{C_r} \frac{\sin(z)}{z(e^z - 1)} \, dz$$

where $C_r$ is the image of $\gamma_r(t) = re^{it}$ for $t \in [0, 2\pi]$, for any $r \in (0, 2\pi)$.

Solution: $z(e^z - 1)$ has a zero of order 2 at the origin, we may compute the residue by using power series

$$\frac{\sin(z)}{z(e^z - 1)} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots}{z^2 \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\right)} = \frac{1 - \frac{z^2}{3!} + \ldots}{z \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\right)}$$

and see that this has a pole of order 1 with residue 1. So for small circles, the value of the integral is $2\pi i$. □

**FQ10**. Suppose that $f(z)$ is holomorphic on a domain $\Omega$. Show that $f'(z)$ is holomorphic on $\Omega$ as well, and show that the following converse is false: every holomorphic function on $\Omega$ is the derivative of a holomorphic function on $\Omega$.

Solution: Holomorphicity is a local condition: There exists a neighborhood on which $f$ is recovered by a power series. This follows from term by term differentiation of the power series. Alternatively, in any neighborhood integrating the derivative of $f'(z)$ around any closed path in the domain will return the value 0, so by Morera’s theorem, $f'(z)$ is analytic. Counterexample $\frac{1}{z}$ is holomorphic on $z \neq 0$, however there is no antiderivative on any punctured disk. □