Here are some definitions and theorems which may or may not be useful for the exam.

- A pair \((X, A)\) is \(n\)-connected if each path component of \(X\) contains a unique path component of \(A\) and \(\pi_i(X, A, x_0) = 0\) for all \(i \leq n\) and \(x_0 \in A\).

- **Hurewicz Theorem.** If \(X\) is \((n - 1)\)-connected, \(n \geq 2\), then \(\tilde{H}_i(X) = 0\) for \(i < n\) and the Hurewicz map \(\pi_n(X) \rightarrow H_n(X)\) is an isomorphism. If \((X, A)\) is \((n - 1)\)-connected, \(n \geq 2\), with \(A\) simply connected and non-empty then \(H_i(X, A) = 0\) for \(i < n\) and the Hurewicz map \(\pi_n(X, A) \rightarrow H_n(X, A)\) is an isomorphism. More generally, for \(A\) not simply connected, \(\pi_n(X, A) \rightarrow H_n(X, A)\) is surjective with kernel the normal subgroup generated by \(\{ \gamma \cdot f - \gamma \} \) for \(\gamma \in \pi_1(A)\) and \(f \in \pi_n(X, A)\) and \(H_i(X, A) = 0\) for \(i < n\).

- **Excision theorem for homotopy groups.** Let \((A, C)\) and \((B, C)\) be CW pairs and \(X = A \cup B\). If \((A, C)\) is \(m\)-connected and \((B, C)\) is \(n\)-connected then the map \(\pi_i(A, C) \rightarrow \pi_i(X, B)\) induced by inclusion is an isomorphism for \(i < m + n\) and a surjection for \(i = m + n\).

- **Freudenthal Suspension Theorem.** If \(X\) is an \((n - 1)\)-connected CW complex then the suspension map \(\pi_i(X) \rightarrow \pi_{i+1}(SX)\) is an isomorphism for \(i < 2n - 1\) and a surjection for \(i = 2n - 1\).
Problems

(1) Let \( X = \{(x, y, z) \in \mathbb{R}^3 \mid xy = 0\} \), which is the union of the \( xz \)- and \( yz \)-planes. Prove that for any neighborhood \( U \subset X \) containing the origin, \( U \setminus 0 \) retracts onto a set with non-abelian fundamental group, so \( \pi_1(U \setminus 0) \) is non-abelian. Use this to show that \( X \) is not a manifold.

(2) Let \( A \) and \( B \) be solid tori, that is, copies of \( D^2 \times S^1 \). Identify \( \partial A \) with the quotient space \( \mathbb{R}^2 / \mathbb{Z}^2 \) of the plane by the action of \( \mathbb{Z}^2 \) by translations, in such a way that the loop \( \gamma \) in \( \partial A \) determined by the path \([0, 1] \to \mathbb{R}^2, t \mapsto (t, 0)\) becomes contractible in \( A \), and the loop \( \eta \) determined by \( t \mapsto (0, t) \) becomes a generator of \( \pi_1(A) \). Do the same with \( \partial B \).

(a) Let \( X \) be the space obtained by gluing \( A \) and \( B \) via the homeomorphism \( \partial A \to \partial B \) determined by the map \( \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (y, x) \). Prove that \( \pi_1(X) = 0 \).

(b) Let \( Y \) be the space obtained by gluing \( A \) and \( B \) via the homeomorphism \( \partial A \to \partial B \) determined by the map \( \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x, x + y) \). Prove that \( \pi_1(Y) = 0 \).

(3) Let \( X \) and \( Y \) be closed, connected oriented surfaces, and let \( f : Y \to X \) be a branched double cover, meaning that there are finitely many points \( x_1, \ldots, x_n \in X \) near which \( f \) looks like the map \( \mathbb{C} \to \mathbb{C} \) given by \( z \mapsto z^2 \), and \( f \) is an honest 2-sheeted cover over the complement \( U := X \setminus \{x_1, \ldots, x_n\} \). (“Looks like” means that for each \( i \) there is a neighborhood \( V \) of \( x_i \), a neighborhood \( W \) of \( y_i = f^{-1}(x_i) \), and homeomorphisms \( \phi : (V, x_i) \to (D^2, 0), \psi : (W, y_i) \to (D^2, 0) \) so that \( f = \phi^{-1} \circ (z \mapsto z^2) \circ \psi \).)

Assume that \( n \geq 1 \). Prove that \( f_* : \pi_1(Y) \to \pi_1(X) \) is surjective.

(4) (a) Give a list of axioms for a (generalized) reduced homology theory.

(b) Define a functor \( \overline{h}_n \) from spaces to abelian groups by

\[ \overline{h}_n(X) = \overline{H}_n(X; \mathbb{Z}) \oplus \text{Hom}_\mathbb{Z}(\mathbb{Z}/2\mathbb{Z}, \overline{H}_{n-1}(X; \mathbb{Z})) \]

and, for \( f : X \to Y \), \( \overline{h}_n(f)(\alpha, \beta) = (f_*(\alpha), f_\circ \beta) \) (where \( (\alpha, \beta) \in \overline{h}_n(X) \) and \( f_* \) is the usual induced map on singular homology). Prove that the \( \overline{h}_n \) are not a (generalized) reduced homology theory.

(5) Let \( S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = 1\} \). Call a continuous map \( f : S^{2n-1} \to Y \) complex-even if \( f(e^{i\theta}z) = f(z) \) for all \( z \in S^{2n-1} \) and \( \theta \in \mathbb{R} \). Show that any complex-even map \( f : S^{2n-1} \to S^{2n-1} \) has a fixed point.

(6) Show that \( \mathbb{RP}^2 \land \mathbb{RP}^2 \) is not homotopy equivalent to a closed manifold.

(7) (a) Define the degree of a map between closed, connected, oriented, non-empty \( n \)-manifolds.

(b) Let \( Y \) be a 3-dimensional lens space. Show that any map \( Y \to S^2 \times S^1 \) has degree 0, for any choice of orientations.

(8) Suppose \( X \) is a closed, connected 4-manifold with \( H_1(X; \mathbb{Z}) = H_2(X; \mathbb{Z}) = 0 \). Prove that \( \Sigma X \simeq S^5 \).

(9) Let \( T^3 = S^1 \times S^1 \times S^1 \) denote the 3-dimensional torus.

(a) Compute \([S^3, T^3] \), the set of homotopy classes of maps from \( S^3 \) to \( T^3 \). That is, say how many elements this set has and give a (relatively) explicit description of (a representative for) each element.

(b) Compute \([T^3, S^3] \), the set of homotopy classes of maps from \( T^3 \) to \( S^3 \). That is, say how many elements this set has and give a (relatively) explicit description of (a representative for) each element.

(10) Is the Eilenberg-MacLane space \( K(\mathbb{Z}/2\mathbb{Z}, 2) \) homotopy equivalent to a finite CW complex? If so, that describe that CW complex explicitly. If not, prove it.