UO TOPOLOGY QUALIFYING EXAM
FALL 2021

Precise statements of selected definitions and theorems

Here are some definitions and theorems which may or may not be useful for the exam.

- A pair (X, A) is *n-connected* if each path component of X contains a unique path component of A and $\pi_i(X, A, x_0) = 0$ for all $i \leq n$ and $x_0 \in A$.
- *Hurewicz Theorem.* If X is $(n-1)$-connected, $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i < n$ and the Hurewicz map $\pi_n(X) \to H_n(X)$ is an isomorphism. If (X, A) is $(n-1)$-connected, $n \geq 2$, with A simply connected and non-empty then $H_i(X, A) = 0$ for $i < n$ and the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ is an isomorphism. More generally, for A not simply connected, $\pi_n(X, A) \to H_n(X, A)$ is surjective with kernel the normal subgroup generated by $\{\gamma \cdot f - \gamma\}$ for $\gamma \in \pi_1(A)$ and $f \in \pi_n(X, A)$ and $H_i(X, A) = 0$ for $i < n$.
- *Excision theorem for homotopy groups.* Let (A, C) and (B, C) be CW pairs and $X = A \cup B$. If (A, C) is m-connected and (B, C) is n-connected then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.
- *Freudenthal Suspension Theorem.* If X is an $(n-1)$-connected CW complex then the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Problems

(1) Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid xy = 0\}$, which is the union of the xz- and yz-planes. Prove that for any neighborhood $U \subset X$ containing the origin, $U \setminus 0$ retracts onto a set with non-abelian fundamental group, so $\pi_1(U \setminus 0)$ is non-abelian. Use this to show that X is not a manifold.

(2) Let A and B be solid tori, that is, copies of $D^2 \times S^1$. Identify ∂A with the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ of the plane by the action of $\mathbb{Z}^2$ by translations, in such a way that the loop γ in ∂A determined by the path $[0, 1] \to \mathbb{R}^2$, $t \mapsto (t, 0)$ becomes contractible in A, and the loop η determined by $t \mapsto (0, t)$ becomes a generator of $\pi_1(A)$. Do the same with ∂B.
  (a) Let X be the space obtained by gluing A and B via the homeomorphism $\partial A \to \partial B$ determined by the map $\mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (y, x)$. Prove that $\pi_1(X) = 0$.
  (b) Let Y be the space obtained by gluing A and B via the homeomorphism $\partial A \to \partial B$ determined by the map $\mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, x + y)$. Prove that $\pi_1(Y) = 0$.

(3) Let X and Y be closed, connected oriented surfaces, and let $f : Y \to X$ be a *branched double cover*, meaning that there are finitely many points $x_1, \dots, x_n \in X$ near which f looks like the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$, and f is an honest 2-sheeted cover over the complement $U := X \setminus \{x_1, \dots, x_n\}$. ("Looks like" means that for each i there is a neighborhood V of x_i, a neighborhood W of $y_i = f^{-1}(x_i)$, and homeomophisms $\phi : (V, x_i) \to (D^2, 0)$, $\psi : (W, y_i) \to (D^2, 0)$ so that $f = \phi^{-1} \circ (z \mapsto z^2) \circ \psi$.)
Assume that $n \geq 1$. Prove that $f_* : \pi_1(Y) \to \pi_1(X)$ is surjective.

(4) (a) Give a list of axioms for a (generalized) reduced homology theory.
  (b) Define a functor $\tilde{h}_n$ from spaces to abelian groups by

$$\tilde{h}_n(X) = \widetilde{H}_n(X; \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \widetilde{H}_{n-1}(X; \mathbb{Z}))$$

and, for $f : X \to Y$, $h_n(f)(\alpha, \beta) = (f_*(\alpha), f_* \circ \beta)$ (where $(\alpha, \beta) \in \tilde{h}_n(X)$ and f_* is the usual induced map on singular homology). Prove that the $\tilde{h}_n$ are not a (generalized) reduced homology theory.

(5) Let $S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = 1\}$. Call a continuous map $f : S^{2n-1} \to Y$ *complex-even* if $f(e^{i\theta} z) = f(z)$ for all $z \in S^{2n-1}$ and $\theta \in \mathbb{R}$. Show that any complex-even map $f : S^{2n-1} \to S^{2n-1}$ has a fixed point.

(6) Show that $\mathbb{R}P^2 \wedge \mathbb{R}P^2$ is not homotopy equivalent to a closed manifold.

(7) (a) Define the degree of a map between closed, connected, oriented, non-empty n-manifolds.
  (b) Let Y be a 3-dimensional lens space. Show that any map $Y \to S^2 \times S^1$ has degree 0, for any choice of orientations.

(8) Suppose X is a closed, connected 4-manifold with $H_1(X; \mathbb{Z}) = H_2(X; \mathbb{Z}) = 0$. Prove that $\Sigma X \simeq S^5$.

(9) Let $T^3 = S^1 \times S^1 \times S^1$ denote the 3-dimensional torus.
  (a) Compute $[S^3, T^3]$, the set of homotopy classes of maps from S^3 to T^3. That is, say how many elements this set has and give a (relatively) explicit description of (a representative for) each element.
  (b) Compute $[T^3, S^3]$, the set of homotopy classes of maps from T^3 to S^3. That is, say how many elements this set has and give a (relatively) explicit description of (a representative for) each element.

(10) Is the Eilenberg-MacLane space $K(\mathbb{Z}/2\mathbb{Z}, 2)$ homotopy equivalent to a finite CW complex? If so, that describe that CW complex explicitly. If not, prove it.