UO TOPOLOGY QUALIFYING EXAM
FALL 2021
SOLUTIONS

(1) Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid xy = 0\}$, which is the union of the xz- and yz-planes. Prove that for any neighborhood $U \subset X$ containing the origin, $U \setminus 0$ retracts onto a set with non-abelian fundamental group, so $\pi_1(U \setminus 0)$ is non-abelian. Use this to show that X is not a manifold.

Solution. Choose an $\epsilon > 0$ such that the ball of radius 2ϵ around the origin is contained in U. Let Y be the intersection of X with the sphere of radius ϵ. Then Y consists of two points joined by four arcs. (See Figure 1.) Contracting one of the arcs, we see that Y is homotopy equivalent to a wedge of three circles, so $\pi_1(Y)$ is the free group on three generators, and in particular is non-abelian. Let $i\colon Y \to U \setminus 0$ be the inclusion, and define a map $r\colon U \setminus 0 \to Y$ by $r(v) = \frac{\epsilon}{|v|}v$; then r is a retraction, meaning that $r \circ i = \mathbb{I}_Y$. Thus the induced maps on π_1 satisfy $r_* \circ i_* = \mathbb{I}_{\pi_1(Y)}$, so i_* is injective, and in particular $\pi_1(U \setminus 0)$ is non-abelian.

If X were a manifold, there would be a neighborhood $V \ni 0$ homeomorphic to an open ball in $\mathbb{R}^n$ for some n, so $V \setminus 0$ would be homeomorphic to $\mathbb{R}^n \setminus 0$, hence homotopy equivalent to S^{n-1}. If $n = 2$ we would have $\pi_1(V \setminus 0) = \mathbb{Z}$; otherwise we would have $\pi_1(V \setminus 0) = 0$. This would contradict the fact that $\pi_1(V \setminus 0)$ is nonabelian, so X is not a manifold.

(2) Let A and B be solid tori, that is, copies of $D^2 \times S^1$. Identify ∂A with the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ of the plane by the action of $\mathbb{Z}^2$ by translations, in such a way that the loop γ in ∂A determined by the path $[0, 1] \to \mathbb{R}^2$, $t \mapsto (t, 0)$ becomes contractible in A, and the loop η determined by $t \mapsto (0, t)$ becomes a generator of $\pi_1(A)$. Do the same with ∂B.

(a) Let X be the space obtained by gluing A and B via the homeomorphism $\partial A \to \partial B$ determined by the map $\mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (y, x)$. Prove that $\pi_1(X) = 0$.

Solution 1. Let $U \supset A$ and $V \supset B$ be open subsets of X which deformation retract to A and B, respectively, and so that $U \cap V \simeq T^2 \times (0, 1)$. Applying Van Kampen's

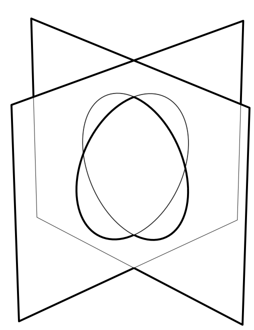


FIGURE 1. **The spaces X and Y.** The space X is the union of the two planes shown, minus the origin. The space Y is the union of the two circles. Hidden lines are thin and gray.

theorm to this open cover, for $x_0 \in A \cap B$ we have

$$\pi_1(X, x_0) = \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) = \pi_1(A, x_0) *_{\pi_1(\partial A, x_0)} \pi_1(B, x_0) \cong \mathbb{Z} *_{\mathbb{Z}^2} \mathbb{Z}.$$

(You would not lose many points by skipping the discussion of U and V.) The group $\pi_1(\partial A, x_0)$ is generated by γ and η. Let η_B be the generator of $\pi_1(B, x_0)$ corresponding to $\eta\colon [0,1] \to \partial B$ and let $i\colon \partial A \to A$ and $j\colon \partial A \to B$ denote the inclusion maps. Then

$$\pi_1(X, x_0) = \langle \eta, \eta_B \mid i_*(\gamma)j_*(\gamma)^{-1},\ i_*(\eta)j_*(\eta)^{-1}\rangle.$$

From the definition of X (and i_* and j_*), $i_*(\gamma) = e$, the identity, and $i_*(\eta) = \eta$, $j_*(\gamma) = \eta_B$, and $j_*(\eta) = e$. So,

$$\pi_1(X, x_0) = \langle \eta, \eta_B \mid \eta_B^{-1},\ \eta\rangle,$$

which is the trivial group.

Solution 2 (sketch). One can show that $X \cong S^3$, as follows. Identify $S^3 = \partial D^4 = \partial(D^2 \times D^2) = D^2 \times S^1 \cup S^1 \times D^2$. Define a map $X \to S^3$ by mapping A to the first $D^2 \times S^1$ by the identity map and sending B to the second $S^1 \times D^2$ by the map that exchanges the factors. Observe that this respects the equivalence relation defining X. Then show that S^3 is simply connected by decomposing it as $D^3 \cup_{S^2} D^3$, say.

(b) Let Y be the space obtained by gluing A and B via the homeomorphism $\partial A \to \partial B$ determined by the map $\mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, x + y)$. Prove that $\pi_1(Y) = 0$.

Solution. With the same setup and notation as in the previous case, we have

$$\pi_1(X, x_0) = \langle \eta, \eta_B \mid i_*(\gamma)j_*(\gamma)^{-1},\ i_*(\eta)j_*(\eta)^{-1}\rangle,$$

where now $i_*(\gamma) = e$, $i_*(\eta) = \eta$, $j_*(\gamma) = \eta_B$, and $j_*(\eta) = \eta_B$. (For the computation of $j_*(\gamma)$, note that $j_*(\gamma) = \gamma_B + \eta_B = \eta_B$.) So,

$$\pi_1(X, x_0) = \langle \eta, \eta_B \mid \eta_B^{-1},\ \eta\eta_B^{-1}\rangle,$$

which again is the trivial group.

Solution 2 (sketch). Again, one can show that $X \cong S^3$, by a variant of the argument in the previous case.

Remark. Other ways of gluing A and B give the 3-dimensional lens spaces $L(p, q)$ and the space $S^2 \times S^1$.

(3) Let X and Y be closed, connected oriented surfaces, and let $f\colon Y \to X$ be a *branched double cover*, meaning that there are finitely many points $x_1, \dots, x_n \in X$ near which f looks like the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$, and f is an honest 2-sheeted cover over the complement $U := X \setminus \{x_1, \dots, x_n\}$. ("Looks like" means that for each i there is a neighborhood V of x_i, a neighborhood W of $y_i = f^{-1}(x_i)$, and homeomophisms $\phi\colon (V, x_i) \to (D^2, 0)$, $\psi\colon (W, y_i) \to (D^2, 0)$ so that $f = \phi^{-1} \circ (z \mapsto z^2) \circ \psi$.)

Assume that $n \geq 1$. Prove that $f_*\colon \pi_1(Y) \to \pi_1(X)$ is surjective.

Solution 1. Let V be a neighborhood of x_1 over which f looks like $z \mapsto z^2$; in particular, we can require that both V and $f^{-1}(V)$ are homeomorphic to open discs. Choose a base point $x \in V \setminus x_1$, and let $f^{-1}(x) = \{y, y'\}$. Choose a path $\alpha\colon I \to f^{-1}(V) \subset Y$ from y' to y; then $f \circ \alpha$ is a loop in X based at x, and it stays in V which is contractible, so $[f \circ \alpha] = 0 \in \pi_1(X, x)$.

Now let $\gamma\colon I \to X$ be a loop based at x. Deform γ to get a homotopic path γ' that avoids the branch points $x_1, \dots, x_n$ and stays in U. Because f is a double cover over U, there is a unique lift $\tilde{\gamma}'\colon I \to f^{-1}(U) \subset Y$ of γ that begins at y and ends at either y or y'.

If $\tilde{\gamma}'$ ends at y, then $[\tilde{\gamma}'] \in \pi_1(Y, y)$ maps to $[\gamma'] = [\gamma] \in \pi_1(X, x)$. If $\tilde{\gamma}'$ ends at y', then the concatenation $\gamma' \cdot \alpha$ is a loop based at y, and $[\gamma' \cdot \alpha]$ maps to $[\gamma'] \cdot [f \circ \alpha] = [\gamma] \cdot 1 \in \pi_1(X, x)$. In either case we see that $[\gamma]$ is in the image of f_*.

Solution 2. Let $y_i = f^{-1}(x_i)$. By Van Kampen's theorem, the map $\pi_1(X \setminus \{x_1, \dots, x_n\}) \to \pi_1(X)$ induced by inclusion is surjective. Choose a basepoint x for X near (but not equal to) x_1 and a basepoint $y \in f^{-1}(x)$ for Y; all fundamental groups below will be computed with respect to these basepoints, but we will suppress them from the notation.

Observe that $X \setminus \{x_1, \dots, x_n\}$ and $Y \setminus \{y_1, \dots, y_n\}$ are path connected. So, since the map

$$f|_{Y \setminus \{y_1, \dots, y_n\}} \colon Y \setminus \{y_1, \dots, y_n\} \to X \setminus \{x_1, \dots, x_n\}$$

is a nontrivial 2-fold cover, $f_*\big(\pi_1(Y \setminus \{y_1, \dots, y_n\})\big)$ is an index 2 subgroup of $\pi_1(X \setminus \{x_1, \dots, x_n\})$. Let γ be a loop around x_1 contained in a disk V around x_1, small enough that $f^{-1}(V)$ is also a disk. Then $f^{-1}(\gamma)$ is a single loop, so γ is not in the image of $f_* \colon \pi_1(Y \setminus \{y_1, \dots, y_n\} \to \pi_1(X \setminus \{x_1, \dots, x_n\})$. Hence, $f_*\big(\pi_1(Y \setminus \{y_1, \dots, y_n\})\big)$ and γ together generate $\pi_1(X \setminus \{x_1, \dots, x_n\})$. Let $i \colon X \setminus \{x_1, \dots, x_n\} \to X$ denote inclusion. Let K be the kernel of i_*; observe that $\gamma \in K$. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(Y \setminus \{y_1, \dots, y_n\}) & \longrightarrow & \pi_1(Y) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \pi_1(X \setminus \{x_1, \dots, x_n\})/K & \underset{i_*}{\twoheadrightarrow} & \pi_1(X). \end{array}$$

The left vertical arrow is surjective, since γ and the image of f_* generate $\pi_1(X \setminus \{x_1, \dots, x_n\})$, and we already observed that the bottom horizontal map is surjective, so the right vertical map must also be surjective, as claimed.

(4) (a) Give a list of axioms for a (generalized) reduced homology theory.

Solution. A *reduced homology theory* consists of functors $\widetilde{h}_n$ from the category of topological spaces and continuous maps [or CW complexes and continuous maps], for $n \in \mathbb{Z}$, so that:

(i) (Homotopy Invariance) If $f, g \colon X \to Y$ are homotopic then $\widetilde{h}_n(f) = \widetilde{h}_n(g)$.

(ii) (Long exact sequence for a pair) Given a pair (X, A) so that the inclusion $A \hookrightarrow X$ satisfies the homotopy extension property there is a map $\partial \colon \widetilde{h}_n(X/A) \to \widetilde{h}_{n-1}(A)$ so that the sequence

$$\cdots \to \widetilde{h}_n(A) \to \widetilde{h}_n(X) \to \widetilde{h}_n(X/A) \xrightarrow{\partial} \widetilde{h}_{n-1}(A) \to \cdots$$

is exact. Here, the other two maps are induced by the inclusion $A \hookrightarrow X$ and the projection $X \to X/A$. Further, ∂ is natural with respect to maps of pairs.

(iii) (Infinite wedge sums) If $X = \bigvee_\alpha X_\alpha$ then for each n the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism $\bigoplus_\alpha \widetilde{h}_n(X_\alpha) \xrightarrow{\cong} \widetilde{h}_n(X)$.

Remarks. There are various other options, including substituting the Mayer-Vietoris sequence in place of the long exact sequence for a pair.

(b) Define a functor $\tilde{h}_n$ from spaces to abelian groups by $\tilde{h}_n(X) = \tilde{H}_n(X) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \tilde{H}_{n-1}(X))$ and, for $f \colon X \to Y$, $h_n(f)(\alpha, \beta) = (f_*(\alpha), f_* \circ \beta)$ (where $(\alpha, \beta) \in \tilde{h}_n(X)$ and f_* is the usual induced map on singular homology). Prove that the $\tilde{h}_n$ are not a (generalized) reduced homology theory.

Solution. Consider the pair $(\mathbb{R}P^2, \mathbb{R}P^1)$. The long exact sequence for this pair would be

$$\cdots \to \widetilde{h}_2(\mathbb{R}P^1) \to \widetilde{h}_2(\mathbb{R}P^2) \to \widetilde{h}_2(S^2) \to \widetilde{h}_1(\mathbb{R}P^1) \to \widetilde{h}_1(\mathbb{R}P^2) \to \widetilde{h}_1(S^2) \to \widetilde{h}_0(\mathbb{R}P^1) \to \cdots .$$

This is

$$\cdots \to 0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to 0 \to \cdots .$$

Since there is no injective map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$, however, this is a contradiction.

(5) Let $S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = 1\}$. Call a continuous map $f\colon S^{2n-1} \to Y$ *complex-even* if $f(e^{i\theta}z) = f(z)$ for all $z \in S^{2n-1}$ and $\theta \in \mathbb{R}$. Show that any complex-even map $f\colon S^{2n-1} \to S^{2n-1}$ has a fixed point.

Solution. Since f is complex-even, f descends to a continuous map $g\colon \mathbb{C}P^{n-1} \to S^{2n-1}$, so that $g \circ p = f$ (where $p\colon S^{2n-1} \to \mathbb{C}P^{n-1}$ is the quotient map). So, $f_*\colon H_{2n-1}(S^{2n-1}) \to H_{2n-1}(S^{2n-1})$ factors as $g_* \circ p_*$, but since $H_{2n-1}(\mathbb{C}P^{n-1}) = 0$, $f_* = g_* \circ p_* = 0$. So, the Lefschetz trace of g is 1 so by the Lefschetz fixed point theorem g has a fixed point.

(6) Show that $\mathbb{R}P^2 \wedge \mathbb{R}P^2$ is not homotopy equivalent to a closed manifold.

Solution. We start by computing the cohomology of $\mathbb{R}P^2 \wedge \mathbb{R}P^2$ with $\mathbb{F}_2$-coefficients. From Hatcher's first version of the Künneth theorem, the map

$$F\colon H^*(\mathbb{R}P^2; \mathbb{F}_2) \otimes H^*(\mathbb{R}P^2; \mathbb{F}_2) \to H^*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{F}_2)$$
$$a \otimes b \mapsto p_1^*(a) \cup p_2^*(b)$$

is an isomorphism. We have $H^*(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^3)$ so $H^*(\mathbb{R}P^2; \mathbb{F}_2) \otimes H^*(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2[x, y]/(x^3, y^3)$. Also, $p_1^*(1) = p_2^*(1) = 1$. Finally, if $i_1, i_2\colon \mathbb{R}P^2 \to \mathbb{R}P^2 \times \mathbb{R}P^2$ are the inclusions as $\mathbb{R}P^2 \times \{pt\}$ and $\{pt\} \times \mathbb{R}P^2$, respectively, then $p_j \circ i_j = \mathbb{I}$. Finally, the two obvious inclusion maps of $\mathbb{R}P^2$ into $\mathbb{R}P^2 \vee \mathbb{R}P^2$ induce an isomorphism $H^m(\mathbb{R}P^2 \vee \mathbb{R}P^2; \mathbb{F}_2) \cong H^m(\mathbb{R}P^2; \mathbb{F}_2) \oplus H^m(\mathbb{R}P^2; \mathbb{F}_2)$ for $m > 1$. It follows that the inclusion map $\iota\colon \mathbb{R}P^2 \vee \mathbb{R}P^2 \hookrightarrow \mathbb{R}P^2 \times \mathbb{R}P^2$ has $\iota^*(1) = 1$, $\iota^*(x) = (x, 0)$, and $\iota^*(y) = (0, y)$.

Now, since ι^* is surjective, the long exact sequence for the pair $(\mathbb{R}P^2 \times \mathbb{R}P^2, \mathbb{R}P^2 \vee \mathbb{R}P^2)$ gives that for $m > 0$,

$$H^m(\mathbb{R}P^2 \vee \mathbb{R}P^2; \mathbb{F}_2) = \ker\bigl(\iota^*\colon H^m(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{F}_2) \to H^m(\mathbb{R}P^2 \vee \mathbb{R}P^2; \mathbb{F}_2)\bigr)$$

So,

$$H^m(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & m = 0 \\ 0 & m = 1 \\ \mathbb{F}_2\langle xy \rangle & m = 2 \\ \mathbb{F}_2\langle x^2y, xy^2 \rangle & m = 3 \\ \mathbb{F}_2\langle x^2y^2 \rangle & m = 4 \\ 0 & m > 4. \end{cases}$$

Since every manifold is $\mathbb{F}_2$-orientable, if $\mathbb{R}P^2 \wedge \mathbb{R}P^2$ is homotopy equivalent to a closed manifold then that manifold must have dimension 4. Then, since $\mathbb{F}_2$ is a field, Poincaré duality would imply that $\dim H^m(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{F}_2) = H^{4-m}(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{F}_2)$, but this fails for $m = 1$.

Solution 2 (sketch). Like solution 1, but using the Künneth theorem for the smash product and reduced homology to shorten the computation of the cohomology of $\mathbb{R}P^2 \wedge \mathbb{R}P^2$.

Solution 2 (sketch). Like solution 1, but using the cellular chain complex for $\mathbb{R}P^2 \wedge \mathbb{R}P^2$ to compute the homology or cohomology.

Solution 3 (sketch). Like solution 1, but using the Künneth theorem for homology instead of cohomology, and the projection $\mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}P^2$ to compute the map induced by the inclusion $\mathbb{R}P^2 \vee \mathbb{R}P^2 \to \mathbb{R}P^2 \times \mathbb{R}P^2$.

(7) (a) Define the degree of a map between closed, connected, oriented, non-empty n-manifolds.

Solution. Given $f\colon M^n \to N^n$, let $[M]$ be the generator for $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ corresponding to the orientation, and let $[N]$ be the generator for $H_n(N;\mathbb{Z}) \cong \mathbb{Z}$ corresponding to the orientation. Then $f_*[M] = k[N]$ for some integer k. The *degree of* f is k.

(b) Let Y be a 3-dimensional lens space. Show that any map $Y \to S^2 \times S^1$ has degree 0, for any choice of orientations.

Solution. By definition, Y is a quotient of S^3 by a free action of $\mathbb{Z}/n\mathbb{Z}$, for some n. In particular, $S^3 \to Y$ is the universal cover, so $\pi_1(Y) \cong \mathbb{Z}/n\mathbb{Z}$. Let $p\colon S^2 \times \mathbb{R} \to S^2 \times S^1$ be the universal cover. Suppose $f\colon Y \to S^2 \times S^1$. Then $f_*\colon \mathbb{Z}/n\mathbb{Z} = \pi_1(Y) \to \pi_1(S^2 \times S^1) = \mathbb{Z}$ vanishes, so f lifts to a map $g\colon Y \to S^2 \times \mathbb{R}$ (so that $f = p \circ g$). Then $f_*[Y] = p_*(g_*[Y]) = p_*(0) = 0$, since $H_3(S^2 \times \mathbb{R}) = 0$. So, f has degree 0.

(8) Suppose X is a closed, connected 4-manifold with $H_1(X) = H_2(X) = 0$. Prove that $\Sigma X \simeq S^5$.

Solution 1. Since $H_1(X) = 0$, X is orientable. So, in particular, $H_4(X) \cong \mathbb{Z}$ and $H_3(X)$ is torsion-free. Also, by a result from the appendix in Hatcher (specifically Corollaries A.8 and A.9), the homology of X is finitely-generated. So, $H_3(X) \cong \mathrm{Hom}(H^3(X), \mathbb{Z})$. By Poincaré duality, $H^3(X) \cong H_1(X) = 0$. To summarize, we have

$$H_i(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, 4 \\ 0 & \text{otherwise.} \end{cases}$$

If we let $B \subset X$ be a closed ball with interior U, then the quotient map $q\colon X \to X/X \setminus U = B/\partial B = S^4$ has degree 1 (consider the local degree at a point in B), so induces an isomorphism on homology.

Now, ΣX is simply connected (by Van Kampen's theorem, say). Then $\Sigma q\colon \Sigma X \to \Sigma S^4 = S^5$ is still an isomorphism on homology. So, by Whitehead's theorem, q is a homotopy equivalence. (Note that, by another result from Hatcher's appendix, X and hence ΣX has the homotopy type of a CW complex.)

Solution 2. Alternatively, at the end of the proof one can appeal to uniqueness of Moore spaces: ΣX is an $M(\mathbb{Z}, 5)$.

(9) (a) Compute $[S^3, T^3]$, the set of homotopy classes of maps from S^3 to T^3. That is, say how many elements this set has and give a (relatively) explicit description of (a representative for) each element.

Solution 1. Since $T^3 = S^1 \times S^1 \times S^1 = K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 1)$, T^3 is a $K(\mathbb{Z}, 3)$, so $[S^3, T^3] \cong H^1(S^3; \mathbb{Z}^3) = \{0\}$. So, there is a unique homotopy class of maps $S^3 \to T^3$. A constant map $S^3 \to T^3$ is a representative.

Solution 2. Equivalently, the universal cover $p\colon \mathbb{R}^3 \to T^3$ is contractible. Since S^3 is simply connected, any map $f\colon S^3 \to T^3$ lifts to a map $g\colon S^3 \to \mathbb{R}^3$, with $p \circ g = f$. The map g is homotopic to a constant map, so $f = p \circ g$ is also homotopic to a constant map. So, there is a unique homotopy class of maps $S^3 \to T^3$, represented by the constant maps.

(b) Compute $[T^3, S^3]$, the set of homotopy classes of maps from T^3 to S^3. That is, say how many elements this set has and give a (relatively) explicit description of (a representative for) each element.
Solution. Build $K(\mathbb{Z}, 3)$ from S^3 by attaching cells of dimension ≥ 5. By cellular approximation, $[T^3, S^3] = [T^3, K(\mathbb{Z}, 3)] \cong H^3(T^3; \mathbb{Z}) \cong \mathbb{Z}$. So, there are countably infinitely many homotopy classes of maps $T^3 \to S^3$. They are distinguished by $f^*[S^3]$, where $[S^3] \in H^3(S^3)$ is a generator.
Consider the map $S^1 \to S^1$ given by $z \mapsto z^n$. Suspending this map twice gives a map $g_n \colon S^3 \to S^3$. Let $f_1 \colon T^3 \to S^3$ be the map which collapses the 2-skeleton of T^3 (with its standard CW structure) to a point. Then the maps $f_n = g_n \circ f_1$, $n \in \mathbb{Z}$, represent the different homotopy classes.

(10) Is the Eilenberg-MacLane space $K(\mathbb{Z}/2\mathbb{Z}, 2)$ homotopy equivalent to a finite CW complex? If so, that describe that CW complex explicitly. If not, prove it.

Solution 1. No, it is not. Recall that elements of $H^i(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{Z}/2\mathbb{Z})$ are in bijection with natural cohomology operations

$$\eta \colon H^2(X; \mathbb{Z}/2\mathbb{Z}) \to H^i(X; \mathbb{Z}/2\mathbb{Z}).$$

We will show that there is a nontrivial operation of this form for each even $i \geq 2$. It follows that $H^{2j}(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{Z}/2\mathbb{Z}) \neq 0$ for all $j \geq 1$. Hence, by cellular homology, any CW structure for $K(\mathbb{Z}/2\mathbb{Z}, 2)$ must have at least one cell in each even dimension.

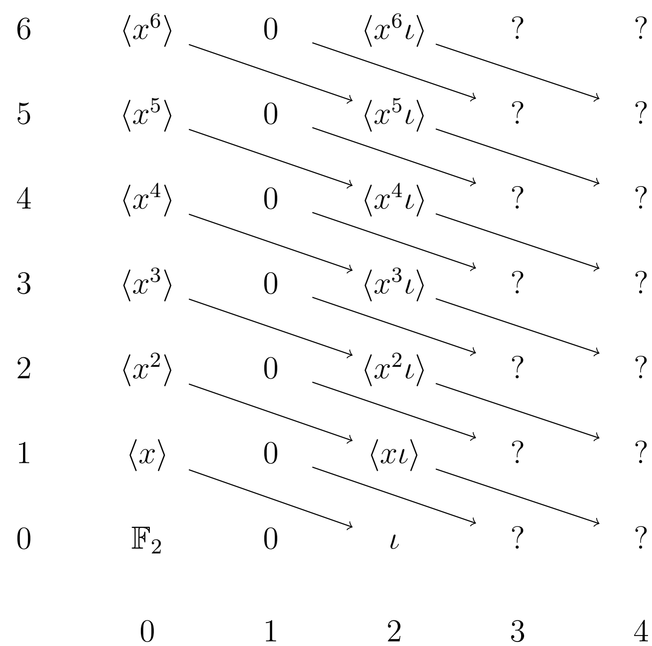
Consider the map

$$\eta_j(\alpha) = \alpha^j = \overbrace{\alpha \cup \cdots \cup \alpha}^{j},$$

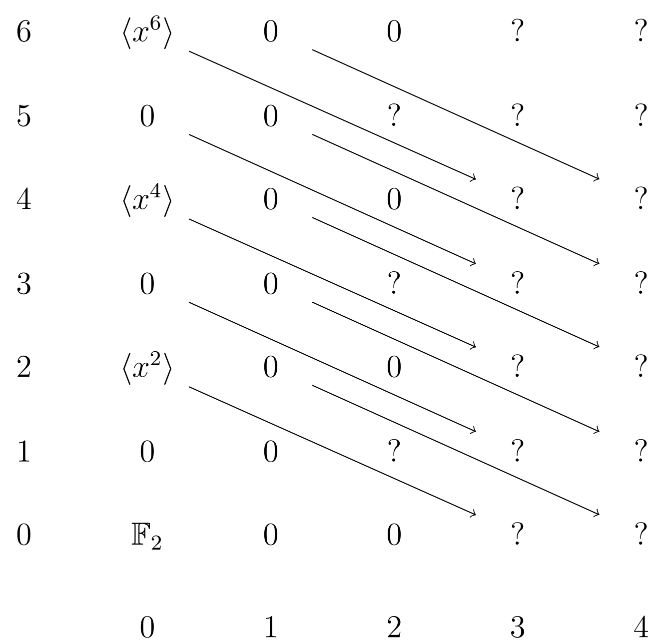
viewed as a map $H^2(X; \mathbb{Z}/2\mathbb{Z}) \to H^{2j}(X; \mathbb{Z}/2\mathbb{Z})$. Naturality of the cup product implies that η_j is natural. On the other hand, for x a generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$, we have $\eta_j(x \cup x) \neq 0$, so η_j is nontrivial. This proves the result.

Solution 2. No, it is not. To see this, by cellular homology it suffices to show that $K(\mathbb{Z}/2\mathbb{Z}, 2)$ has nontrivial cohomology in infinitely many gradings, with some coefficients.

Recall that $H^*(K(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{F}_2) = H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$ where $x \in H^1$. Now, consider the path-loop fibration $K(\mathbb{Z}/2\mathbb{Z}, 1) \to PK(\mathbb{Z}/2\mathbb{Z}, 2) \to K(\mathbb{Z}/2\mathbb{Z}, 2)$. The Serre spectral sequence with $\mathbb{F}_2$-coefficients has the form

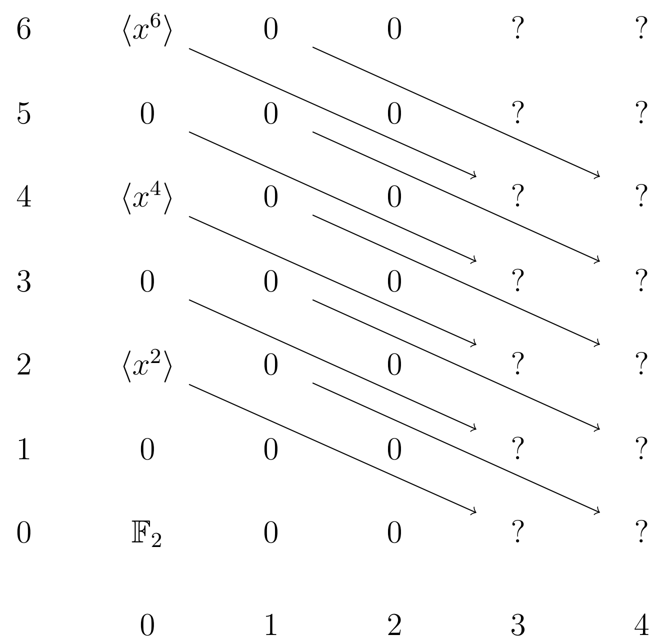


Since the path space is contractible, the E^∞-term must vanish (except for the $\mathbb{F}_2$ in bigrading $(0,0)$). This is the last opportunity for x to cancel so the d_2-differential must send x to ι. Then by the Leibniz rule, $d_2(x^n) = x^{n-1}\iota$ if n is odd and 0 if n is even. So, the E^3-page has the form



We acquired some new question marks because we don't know *a priori* if $d_2(x\iota) = \iota^2$ is zero or not; but in fact that was the last opportunity for $x\iota$ to cancel, so $d_2(x\iota) = \iota^2$

must be non-zero, and the E_3-page really has the form



Now, there must be a class for x^2 to cancel with; call it ι_2. Then

$$d_3(x^{2n}) = \begin{cases} x^{2n-2}\iota_2 & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

So, on the E^4-page, the classes x^{4n} survive. In particular, the first column is now $\mathbb{F}_2[x^4]$.

The pattern repeats: at some later page, x^4 cancels with something farther to the right than what x^2 canceled with, but then x^{8n} survives and, in fact, the first column is $\mathbb{F}_2[x^8]$, and so on. Thus, $H^n(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{F}_2)$ must be nontrivial in infinitely many gradings, so $K(\mathbb{Z}/2\mathbb{Z}, 2)$ is not a finite CW complex.