Probability Qualifying Exam 2021

There are eight problems on this test. Read each problem carefully before beginning. PARTIAL CREDIT CANNOT BE AWARDED UNLESS YOUR WORK IS CLEAR.

Problem	Possible Points	Earned Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

Problem 1. Let $a \wedge b := \min\{a, b\}$.

For random variables X, Y (defined on the same probability space), let

$$\rho(X, Y) := \mathbb{E}(|X - Y| \wedge 1)\,.$$

For sequences $\{(X_n, Y_n)\}$, where X_n and Y_n are defined on the same probability space, show that $\lim_{n\to\infty} \rho(X_n, Y_n) = 0$ if and only if $X_n - Y_n \to 0$ **in probability**.

Solution: 5 points for each direction.

Recall that $X_n - Y_n \to 0$ in probability means, by definition, that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - Y_n| > \varepsilon) = 0\,.$$

Suppose that $X_n - Y_n \to 0$ in probability. For any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}(|X_n - Y_n| \wedge 1) &= \int_{|X_n - Y_n| < \varepsilon} (|X_n - Y_n| \wedge 1)\, d\mathbb{P} + \int_{|X_n - Y_n| \geq \varepsilon} (|X_n - Y_n| \wedge 1)\, d\mathbb{P} \\ &\leq \varepsilon + \mathbb{P}(|X_n - Y_n| \geq \varepsilon)\,.\end{aligned}$$

For n large enough, by definition of convergence in probability, the second term on the right can be made less than ε. Thus

$$\rho(X_n, Y_n) \leq 2\varepsilon$$

for n sufficiently large. Since ε is arbitrary, $\lim_n \rho(X_n, Y_n) = 0$.

Suppose on the other hand that $\rho(X_n, Y_n) \to 0$ as $n \to \infty$.

For any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}(|X_n - Y_n| \wedge 1) &= \int_{|X_n - Y_n| \geq 1} (|X_n - Y_n| \wedge 1) d\mathbb{P} + \int_{\varepsilon < |X_n - Y_n| < 1} (|X_n - Y_n| \wedge 1) d\mathbb{P} \\ &\quad + \int_{|X_n - Y_n| \leq \varepsilon} (|X_n - Y_n| \wedge 1) d\mathbb{P} \\ &\geq \mathbb{P}(|X_n - Y_n| > 1) + \varepsilon\mathbb{P}(\varepsilon < |X_n - Y_n| < 1) + 0\,.\end{aligned}$$

Thus both terms on the right go to 0 as $n \to \infty$, which implies

$$\mathbb{P}(|X_n - Y_n| > 1) + \mathbb{P}(\varepsilon < |X_n - Y_n| < 1) = \mathbb{P}(|X_n - Y_n| > \varepsilon) \to 0\,.$$

Problem 2. Suppose that $X_1, X_2, \ldots$ are random variables satisfying $\mathbb{E}(X_i) = 0$ and such that $\mathbb{E}(X_i^2) < \infty$ for all i. Assume further that there is a constant $C_1 > 0$ (not depending on i or j) so that

$$\mathbb{E}[X_i X_j] \le \frac{C_1}{|i-j|^2 + 1}.$$

Let $S_n = \sum_{i=1}^n X_i$.

(a) Show that $\mathrm{Var}(S_{n+m} - S_n) \le C_2 m$ for all n, m, where C_2 is a constant (not depending on n, m).

(b) Use (a) to show that $\lim_{n\to\infty} \frac{S_{n^2}}{n^2} = 0$ a.s.

(c) Show that $\lim_{n\to\infty} \frac{S_n}{n} = 0$ a.s.

Hint: Show that the events $A_n = \{\bigcup_{m \le 2n} |S_{n^2+m} - S_{n^2}| > \varepsilon n^2\}$ occur only finitely often.

Solution:

We have

$$\begin{aligned}
\mathrm{Var}(S_{n+m} - S_n) &= \mathbb{E}\left[\left(\sum_{k=1}^m X_{n+k}\right)^2\right] \\
&\le 2\sum_{k=1}^m \sum_{j \le k} \mathbb{E}[X_{n+j} X_{n+k}] \\
&\le 2\sum_{k=1}^m \sum_{j \le k} \frac{C_1}{(k-j)^2 + 1} \\
&\le 2\sum_{k=1}^m \sum_{\ell=0}^\infty \frac{C_1}{\ell^2 + 1} \\
&\le C_2 m.
\end{aligned}$$

This establishes the first inequality.

Now

$$\begin{aligned}
\mathbb{P}(|S_{n^2}| > n^2 \varepsilon) &\le \frac{\mathrm{Var}(S_{n^2})}{n^4 \varepsilon^2} \\
&\le \frac{C_2 n^2}{\varepsilon^2 n^4} = \frac{C_2}{\varepsilon^2 n^2}.
\end{aligned}$$

Thus, the first Borel-Cantelli Lemma shows that

$$\mathbb{P}\{|S_{n^2}| > \varepsilon n^2 \text{ i.o.}\} = 0.$$

Consequently,

$$\mathbb{P}\left(\bigcup_{\varepsilon\in\mathbb{Q}^+}\bigcap_{N}\bigcup_{n\geq N}\{|S_{n^2}/n^2|>\varepsilon\}\right)=0\,.$$

In other words, a.s.,

$$\lim_{n\to\infty}\frac{S_{n^2}}{n^2}=0\,.$$

For the last part, for $m\leq 2n$,

$$\mathbb{P}\{|S_{n^2+m}-S_{n^2}|>\varepsilon n^2\}\leq\frac{\mathrm{Var}(S_{n^2+m}-S_{n^2})}{\varepsilon^2 n^4}\leq\frac{C_2 m}{\varepsilon^2 n^4}\leq\frac{C_3}{\varepsilon^2 n^3}\,.$$

Using a union bound over the $2n$ events in the union,

$$\mathbb{P}\left\{\bigcup_{m\leq 2n}|S_{n^2+m}-S_{n^2}|>\varepsilon n^2\right\}\leq\frac{C_3 2n}{\varepsilon^2 n^3}\leq\frac{C_4}{\varepsilon^2 n^2}\,.$$

Applying Borel-Cantelli again, we have that, for

$$A_n(\varepsilon):=\bigcup_{m\leq 2n}\{|S_{n^2+m}-S_{n^2}|>\varepsilon n^2\}\,,$$

since $\sum_n\mathbb{P}(A_n(\varepsilon))\leq\sum_n\frac{C_4}{\varepsilon^2 n^2}<\infty$,

$$\mathbb{P}(A_n\ \text{i.o.})=0\,.$$

Suppose that $\omega\in G=\{A_n(\varepsilon)\ \text{i.o.}\}^c\cap\{\lim_n S_{n^2}/n^2=0\}$. By the preceding, we know that $\mathbb{P}(G)=1$.

Thus there is some $N(\omega)$ so that for $n>N(\omega)$

$$\max_{m\leq 2n}\frac{|S_{n^2+m}(\omega)-S_{n^2}(\omega)|}{n^2}\leq\varepsilon\quad\text{and}\quad\frac{|S_{n^2}(\omega)|}{n^2}<\varepsilon\,.$$

Thus, for $n>N(\omega)$, for all k satisfying $n^2\leq k<(n+1)^2$,

$$\frac{|S_k(\omega)|}{k}\leq\frac{|S_k(\omega)|}{n^2}\leq\frac{|S_k(\omega)-S_{n^2}(\omega)|}{n^2}+\frac{|S_{n^2}(\omega)|}{n^2}\leq 2\varepsilon\,.$$

In other words,

$$\limsup_{k\to\infty}\frac{|S_k(\omega)|}{k}\leq 2\varepsilon\,.$$

Thus proves that

$$\mathbb{P}\left(\limsup_{k\to\infty}|S_k|/k>2\varepsilon\right)=0\,.$$

Taking a union over positive rational ε, we have

$$\lim_{n\to\infty}\frac{S_n}{n}=0,\quad\text{a.s.}\,.$$

Problem 3. Show that if $\{X_{n,i}\}$ are i.i.d. with $X_{n,i} \in \{0,1\}$ and $\mathbb{E}(X_{n,i}) = p_n$, and $np_i \to \lambda$, then $S_n = \sum_{i=1}^n X_{n,i}$ converges in distribution to a Poisson(λ) distribution.

Solution:

We have, for ϕ_n the fourier transform of S_n,

$$\begin{aligned}
\phi_n(t) &= \mathbb{E}\left[\exp\left(it\sum_{i=1}^n X_{n,i}\right)\right] \\
&= \prod_{i=1}^n \mathbb{E}\left[\exp\left(itX_{n,i}\right)\right] \\
&= \left(\mathbb{E}\left[\exp\left(itX_{n,1}\right)\right]\right)^n \\
&= \left((1-p_n) + p_n e^{it}\right)^n \\
&= (1 - p_n(1-e^{it}))^n \\
&= \left(1 - \frac{np_n(1-e^{it})}{n}\right)^n.
\end{aligned}$$

The second equality follows from independence of $\{X_{n,i}\}_{i=1}^n$. The third equality follows since $\{X_{n,i}\}_{i=1}^n$ are identically distributed. Since $np_n \to \lambda$, we have that

$$\phi_n(t) \to e^{-\lambda(1-e^{it})} = e^{\lambda(e^{it}-1)}.$$

Note the right-hand side is the fourier transform of a Poisson(λ) random variable X:

$$\phi(t) = \mathbb{E}[e^{itX}] = \sum_{k=0}^{\infty} e^{itk}\frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^{it})^k}{k!} = e^{-\lambda(1+e^{it})}.$$

Thus the continuity theorem implies that the distributions converge.

Problem 4. Let $\{Z_{n,k}\}$ be i.i.d. non-negative, integer-valued random variables with $\mathbb{E}(Z_{n,k}) = \mu$. Define, recursively,

$$X_n = \sum_{k=1}^{X_{n-1}} Z_{n,k}.$$

Note $\{X_n\}$ is a branching process with average number of offspring per individual equal to μ.

Show that $\frac{X_n}{\mu^n}$ converges to a finite limit (possibly random) as $n \to \infty$ almost surely.

Solution:

Let $\mathscr{F}_n = \sigma(Z_{m,k} : m \le n)$. Note that

$$\begin{aligned}
\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] &= \mathbb{E}\left[\sum_{k=1}^{X_n} Z_{n+1,k} \,\middle|\, \mathscr{F}_n \right] \\
&= \mathbb{E}\left[\sum_{k=1}^{\infty} Z_{n+1,k} I\{k \le X_n\} \,\middle|\, \mathscr{F}_n \right] \\
&= \sum_{k=1}^{\infty} I\{k \le X_n\} \mathbb{E}[Z_{n+1,k} \mid \mathscr{F}_n].
\end{aligned}$$

This follows since $I\{k \le X_n\}$ is $\mathscr{F}_n$-measurable, so it can be factored outside the conditional expectation. Since $Z_{n+1,k}$ is independent of $\mathscr{F}_n$, we have

$$\mathbb{E}[Z_{n+1,k} \mid \mathscr{F}_n] = \mathbb{E}[Z_{n+1,k}] = \mu,$$

and substituting in we have

$$\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = \sum_{k=1}^{\infty} I\{k \le X_n\}\mu = X_n \mu.$$

We then can find that

$$\mathbb{E}\left[\frac{X_{n+1}}{\mu^{n+1}} \,\middle|\, \mathscr{F}_n \right] = \frac{1}{\mu^{n+1}} \mu X_n = \frac{X_n}{\mu^n}.$$

Thus $M_n = X_n / \mu^n$ defines a martingale. It is non-negative and $\mathbb{E}[M_n] = 1$ for all n, whence it is bounded in L^1. The Martingale Convergence Theorem implies that there exists a finite valued r.v. M_∞ such that $M_n \to M_\infty$ a.s.

Problem 5. Suppose that $\{X_n\}$ is the Markov chain on $\{0,1,\ldots\}$ which satisfies, for $k \ge 1$,

$$P(k,k+1) = 1 - P(k,k-1) = p < \frac{1}{2},$$

and $P(0,0) = 1-p = 1-P(0,1)$. (Informally, $\{X_n\}$ is a nearest-neighbor walk on $\mathbb{N}$ which moves up with probability p and down with probability $1-p$.)

Show that $\mathbb{E}_0(\tau_0) < \infty$, where

$$\tau_0 = \min\{n \ge 1 : X_n = 0\}.$$

Solution: Note that if $\mu(k) = (p/q)^k$, then for $k \ge 1$,

$$\begin{aligned}\mu(k-1)P(k-1,k) + \mu(k+1)P(k+1,k) &= (p/q)^{k-1}p + (p/q)^{k+1}q \\ &= (p/q)^k[q+p] \\ &= \mu(k).\end{aligned}$$

If $k = 0$,

$$\mu(0)P(0,0) + \mu(1)P(1,0) = 1(1-p) + \frac{p}{q}q = 1-p+p = \mu(0).$$

That is, μ satisfies

$$\mu = \mu P,$$

and is thus a stationary measure.

Since $p/q < 1$, we have $\sum_k \mu(k) < \infty$, so it can be normalized to obtain a probability distribution π which satisfies $\pi = \pi P$.

We know that any irreducible chain having such a stationary distribution must be positive recurrent, with

$$\mathbb{E}_0\tau_0 = \frac{1}{\pi(0)} < \infty.$$

Problem 6. Let $(B_t)_{t\geq 0}$ be standard Brownian motion with $B_0 = 0$. Let

$$\tau := \inf\{t \geq 0 : B_t \notin (a,b)\}, \qquad \text{for } a < 0 < b.$$

Let

$$M_t = B_t^3 - 3\int_0^t B_u du.$$

Show that $(M_t)_{t\geq 0}$ is a martingale, and find an expression (in terms of a, b only) for $\mathbb{E}\left[\int_0^\tau B_u du\right]$.

You may find useful that $\mathbb{P}(B_\tau = a) = b/(b-a)$, and may use the fact that $\mathbb{E}[\tau] < \infty$ without proof. State carefully any optional stopping theorem you apply, and show the conditions are satisfied.

Solution:

Applying Itô's formula to $f(x) = x^3$,

$$B_t^3 = B_0^3 + 3\int_0^t B_u^2 dB_u + 3\int_0^t B_u du, \tag{1}$$

and so

$$M_t = 3\int_0^t B_u^2 dB_u. \tag{2}$$

As an integral against dB_u, the process $\{M_t\}$ is a local martingale. In fact it is a martingale, because $\mathbb{E}[\int_0^t B_s^6 ds] < \infty$ (see Theorem 7.6.4 in Durrett).

It now follows by the (version we're using of) the Optional Sampling Theorem that $M_{t\wedge\tau}$ is a martingale, and so

$$\mathbb{E}[M_{\tau\wedge t}] = \mathbb{E}[M_0] = 0.$$

Therefore, for each t,

$$\mathbb{E}[B_{\tau\wedge t}^3] = 3\mathbb{E}[\int_0^{\tau\wedge t} B_u du].$$

The left-hand side is bounded by $(a \vee b)^3$, so by Bounded Convergence, convgerges as $t \to \infty$ to $\mathbb{E}[B_\tau^3]$. The right-hand side is bounded by $\tau(a \vee b)$, a random variable with finite expectation, so by Dominated Convergence, it converges to $\mathbb{E}[\int_0^\tau B_u du$.

Therefore, since $\mathbb{P}\{B_\tau = a\} = b/(b-a)$,

$$\mathbb{E}\left[3\int_0^\tau B_u du\right] = \frac{ba^3 - ab^3}{b-a} \tag{3}$$

$$= ab\frac{a^2 - b^2}{b-a} \tag{4}$$

$$= ab(a+b). \tag{5}$$

Problem 7. Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain on an infinite state space S with transition matrix P, and for an arbitrary subset F of S let $\tau_F = \inf\{t \geq 0 : X_n \in F\}$ be the first hitting time of F. Fix **finite** disjoint subsets A and B of S and let $\mathscr{H}$ be the set of functions $h : S \to \mathbb{R}$ that solve the following system of equations:

$$\begin{aligned} Ph(x) = h(x) \qquad & \text{for } x \notin A \cup B \\ h(x) = 1 \qquad & \text{for } x \in A \\ h(x) = 0 \qquad & \text{for } x \in B. \end{aligned}$$

(a) Show that $h_{A,B}(x) = \mathbb{P}_x\{\tau_A < \tau_B\}$ is in $\mathscr{H}$.

(b) Show that if the chain is transient, then $h_{A,B}$ is *not* the only member of $\mathscr{H}$ by explicitly constructing other solutions.

Hint: consider the function $e(x) = \mathbb{P}_x\{\tau_{A\cup B} = \infty\}$.

Solution:

(a) Clearly, $h_{A,B}$ has the right boundary conditions. By conditioning on the first step, and using the Markov property, if $x \notin A \cup B$,

$$\begin{aligned} h_{A,B}(x) &= \sum_y \mathbb{P}_x\{\tau_A < \tau_B \;\&\; X_1 = y\} \\ &= \sum_y \mathbb{P}_x\{X_1 = y\}\mathbb{P}_x\{\tau_A < \tau_B \parallel X_1 = y\} \\ &= \sum_y p(x,y)\mathbb{P}_y\{\tau_A < \tau_B\} \\ &= Ph_{A,B}(x). \end{aligned}$$

(b) The same argument shows that e solves the same equation except that $e(x) = 0$ for $x \in A$, and so by linearity, for any $\alpha \in \mathbb{R}$, the function $h(x) = h_{A,B}(x) + \alpha e(x)$ is also a solution. Since the chain is transient, $e(x) > 0$ for at least some $x \notin A \cup B$.

Problem 8. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, and define $W_t = tB_{1/t}$. Show that $(W_t)_{t\geq 0}$ is also a standard Brownian motion.

Solution:

Brownian motion is a Gaussian process, and so by sufficiency of means and covariances for determining the distribution of Gaussian processes, it suffices to show that $W_0 = 0$, that for any $t_1 < \cdots < t_n$ that $(W_{t_1}, \ldots, W_{t_n})$ is multivariate Gaussian and that $\mathbb{E}[W_s W_t] = \min(s, t)$.

First, $W_0 = \lim_{t\to\infty} B_t/t$, which is zero almost surely since $\mathbb{E}[B_t/t] = 0$ and $\mathbb{E}[(B_t/t)^2] = 1/t \to 0$.

Next, note that $(W_{t_1}, \ldots, W_{t_n})$ is multivariate Gaussian because it is a linear transformation of $(B_{1/t_1}, \ldots, B_{1/t_n})$.

Finally, compute that if $s < t$,

$$\begin{aligned}
\mathbb{E}[W_s W_t] &= st\mathbb{E}[B_{1/s} B_{1/t}] \\
&= st\min(1/s, 1/t) \\
&= s,
\end{aligned}$$

as required.