Qualifying Exam, Answer Key
Differential Geometry, Fall 2021

F.Q1. Let S^2 denote the unit sphere in $\mathbb{R}^3$ which has standard coordinates (x, y, z). Show that the map

$$f : S^2 \to \mathbb{R}^3, \qquad f(x, y, z) = ((1 - z^2)x, (1 - z^2)y, z)$$

is a smooth one-to-one map but not a smooth embedding.

F.Q1 Sol. (i) f is one-to-one. Suppose

$$((1 - z_1^2)x_1, (1 - z_1^2)y_1, z_1) = f(x_1, y_1, z_1) = f(x_2, y_2, z_2) = ((1 - z_2^2)x_2, (1 - z_2^2)y_2, z_2),$$

then $z_1 = z_2$.
Case (ia) If $1 - z_1^2 = 1 - z_2^2 \neq 0$, then we see $(x_1, y_1) = (x_2, y_2)$ from the equation above.
Case (ib) If $1 - z_1^2 = 1 - z_2^2 = 0$, then we have $0 = (x_1, y_1) = (x_2, y_2)$.
In either case we have proved $(x_1, y_1, z_1) = (x_2, y_2, z_2)$.
(ii) f is smooth. Let $S_{z,+} = \{(x, y, z) \in S^2,\ z > 0\}$ and $\varphi_{z,+} : S_{z,+} \to \mathbb{R}^2$ with $\varphi_{z,+}(x, y) = (x, y)$. Then $(S_{z,+}, \varphi_{z,+}, (x, y))$ and 5 similar others form a coordinate atlas on S^2. We just verify the smoothness of $f \circ \varphi_{z,+}^{-1}$ for example. We have

$$f \circ \varphi_{z,+}^{-1}(x, y) = ((x^2 + y^2)x, (x^2 + y^2)y, \sqrt{1 - x^2 - y^2})$$

which is clearly a smooth function on $\{(x, y),\ x^2 + y^2 < 1\}$.
(iii) f is not an embedding. The Jacobian of $f \circ \varphi_{x,+}^{-1}(x, y)$ at $(0, 0)$ is 0 matrix. □

F.Q2. Let g be a **continuous** Riemannian metric tensor on a smooth manifold M^n (not-necessarily-compact). Show that there is a **smooth** Riemannian metric tensor h on M such that bilinear map

$$f(p) - h(p) : T_pM \times T_pM \to \mathbb{R}$$

is positive definite for all $p \in M$.

F.Q2 Sol. We may choose a (countable) coordinate atlas $\{(U_i, \varphi_i)\}$ of M such that $\varphi_i(U_i) = B(0; 1) \subset \mathbb{R}^n$ (the unit ball) and that there is a smooth partition of unity $\{\rho_i\}$ on M with compact support $\mathrm{supp}(\rho_i) \subset U_i$.

Since g is continuous and positive definite on compact set $\mathrm{supp}(\rho_i)$, there is a small constant $\epsilon_i > 0$ such that $g - \epsilon_i \cdot \varphi_i^* g_{\mathrm{Euc}}$ is positive definite on $\mathrm{supp}(\rho_i)$. Define

$$h = \sum_i \rho_i \cdot \epsilon_i \varphi_i^* g_{\mathrm{Euc}}.$$

It is clear that h is a smooth Riemannian metric on M. Since

$$f(p) - h(p) = \sum_i \rho_i(p) \cdot (f(p) - \epsilon_i \varphi_i^* g_{\mathrm{Euc}}(p)),$$

each term in the summation is non-negative, for the i_0 such that $\rho_{i_0}(p) > 0$ we have the i_0-th term is actually positive definite. □

F.Q3. Construct an explicit diffeomorphism from $\mathbb{R}^3 \setminus \bar{E}$ to $S^2 \times \mathbb{R}$. Here S^2 is round sphere of radius 1 and $\bar{E}$ is closed ellipsoid

$$\bar{E} = \{(x, y, z) \in \mathbb{R}^3,\ \frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{5^2} \leq 1\}.$$

F.Q3 Sol. Let $\bar{B}(0;1) \subset \mathbb{R}^n$ be the closed ball of radius 1 and center 0. We define map

$$\Phi_1 : \mathbb{R}^3 \setminus \bar{E} \to \mathbb{R}^3 \setminus \bar{B}(0;1), \qquad \Phi_1(x, y, z) = \left(\frac{x}{3}, \frac{y}{4}, \frac{z}{5}\right).$$

Clearly Φ_1 is a diffeomorphism.

Next we define a map

$$\Phi_2 : \mathbb{R}^3 \setminus \bar{B}(0;1) \to S^2 \times \mathbb{R}, \quad \Phi_1(x, y, z) = \left(\frac{1}{x^2 + y^2 + z^2}(x, y, z), \ln(x^2 + y^2 + z^2 - 1)\right).$$

It is easy to check that Φ_2 is a diffeomorphism. The required diffeomorphism is given by $\Phi_2 \circ \Phi_1$. $\square$

F.Q4. A smooth real-valued function f, defined on some open subset $U \subset \mathbb{R}^n$, is called **harmonic** if $\sum_{i=1}^n \frac{\partial^2 f}{(\partial x^i)^2} = 0$ on U. Show that f is harmonic if and only if for every $p \in U$ and every positive number r with closed ball $\bar{B}(p;r) \subset U$ of radius r and center p, we have

$$\sum_{i=1}^n (-1)^i \int_{S(p;r)} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n = 0,$$

where $S(p;r) = \partial \bar{B}(p;r)$ is the boundary, and $\widehat{dx^i}$ indicates that dx^i is omitted from the wedge product.

F.Q4 Sol. We compute by using Stokes theorem

$$\begin{aligned}
&\sum_{i=1}^n (-1)^i \int_{S(p;r)} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \int_{B(p;r)} d\left(\sum_{i=1}^n (-1)^i \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n\right) \\
&= \int_{B(p;r)} -\left(\sum_{i=1}^n \frac{\partial^2 f}{(\partial x^i)^2}\right) dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n.
\end{aligned}$$

$\Longrightarrow$ This is clear since the last term in the identity above is zero.

$\Longleftarrow$ Since the first term in the identity above is zero, we have for a fixed $p \in U$

$$\begin{aligned}
&-\left(\sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x^i)^2}\right)(p) \\
=&\lim_{r\to 0^+} \frac{1}{\mathrm{Vol(B(p;r))}} \int_{B(p;r)} -\left(\sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x^i)^2}\right) dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n \\
=&\lim_{r\to 0^+} \frac{1}{\mathrm{Vol(B(p;r))}} \cdot 0 \\
=&0.
\end{aligned}$$

□

F.Q5. Let M^n be an oriented closed manifold and let $U \subset M$ be an open set. Fix a smooth n-form ω on M with $\int_M \omega = 1$.

(F.Q5a) Prove that there is a smooth n-form ω_1 with support $\operatorname{supp}\omega_1 \subset U$ such that $\omega - \omega_1 = d\eta$ for some $(n-1)$-form η on M.

(F.Q5b) Let N^n be a closed manifold with an orientation and let $F : N \to M$ be a smooth map such that $q \in M$ is a regular value with $F^{-1}(q) = \{p_1, \cdots, p_k\}$. Porve that $|\int_N F^*\omega| \leq k$.

F.Q5 Sol. (F.Q5a) We may choose a coordinate chart (V, φ, x) such that $V \subset U$ and $\varphi(V) = B(0;2)$, the unit ball in $\mathbb{R}^n$ centered at 0. Let η be a cutoff function on $B(0;2)$ which equals to 1 on $B(0;1)$. Then we define $\tilde{\omega}_1 \doteqdot \varphi^*(\eta dx^1 \wedge \cdots \wedge dx^n)$ which is a n-form on M. It is clear that $\int_M \tilde{\omega}_1 \neq 0$.

We define $\omega_1 \doteqdot \frac{1}{\int_M \tilde{\omega}_1}\tilde{\omega}_1$. Then $\int_M \omega_1 = 1$ and the deRham cohomology class $[\omega] = [\omega_1] \in H^n_{\mathrm{deR}}(M)$. Hence there is a $(n-1)$-form η on M such that $\omega - \omega_1 = d\eta$. Note that $\operatorname{supp}\omega_1 \subset U$.

(F.Q5b) Since q is a regular value, there is a small neighborhood U of q in M, and neighborhoods W_i of p_i in N such that $F|_{W_i} : W_i \to U$ is a diffeomorphism and $F(N \setminus (\cup_{i=1}^k W_i)) \cap U = \emptyset$. Choose a ω_1 for this U as in (W.Q5a), note that $\operatorname{supp} F^*\omega_1 \subset \cup_{i=1}^k W_i$. We compute by using stokes theorem

$$\begin{aligned}
\left|\int_N F^*\omega\right| &= \left|\int_N F^*\omega_1 + \int_N dF^*\eta\right| = \left|\int_{\cup_{i=1}^k W_i} F^*\omega_1\right| \\
&= \left|\sum_{i=1}^{k} \int_{W_i} F^*\omega_1\right| = \left|\sum_{i=1}^{k} \pm \int_U \omega_1\right|,
\end{aligned}$$

where the sign $\pm$ is determined by whether $F|_{W_i}$ is orientation-preserving and the last equality is got by using $F|_{W_i}$ is a diffeomorphism. The upper bound k now follows from $\int_U \omega_1 = \int_M \omega_1 = 1$. □

F.Q6. If $\eta = \eta_i dx^i$ is a 1-form on some Riemannian manifold (M^n, g), let $\eta_{i,jk} dx^i \otimes dx^j \otimes dx^k$

be the local expression for $\nabla^2\eta$. Prove the Ricci identity

$$\eta_{i,jk} - \eta_{i,kj} = R_{jki}{}^l \eta_l.$$

F.Q6 Sol. Let (x^i) be normal coordinates centered at point p. We compute at p where $\Gamma_{ij}^k(p) = 0$

$$\begin{aligned}\eta_{i,jk} =& \left(\partial_j \eta_i - \Gamma_{ij}^l \eta_l\right)_k \\ =& \partial_k(\partial_j \eta_i - \Gamma_{ij}^l \eta_l) - \Gamma_{ik}^p(\partial_j \eta_p - \Gamma_{pj}^l \eta_l) - \Gamma_{jk}^p(\partial_p \eta_i - \Gamma_{ip}^l \eta_l) \\ =& \partial_k \partial_j \eta_i - (\partial_k \Gamma_{ij}^l)\eta_l.\end{aligned}$$

By symmetry we have

$$\eta_{i,kj} = \partial_j \partial_k \eta_i - (\partial_j \Gamma_{ik}^l)\eta_l.$$

Hence $\eta_{i,jk} - \eta_{i,kj} = -(\partial_k \Gamma_{ij}^l)\eta_l + (\partial_j \Gamma_{ik}^l)\eta_l = R_{jki}{}^l \eta_l$. □

F.Q7. Let $\pi : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ denote the standard projection. For each $t \in [0,1]$, consider the map

$$i_t : \mathbb{R}^n \to \mathbb{R}^n \times [0,1], \qquad i_t(x) = (x,t),$$

where $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$. If ω is a smooth k-form in $\Omega^k(\mathbb{R}^n \times [0,1])$, then obviously ω can be written uniquely as

$$\omega = \tilde{\omega} + dt \wedge \eta$$

where

$$\tilde{\omega} = \sum_{i_1 < \cdots < i_k} \tilde{\omega}_{i_1, \cdots, i_k}(x,t) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and

$$\eta = \sum_{j_1 < \cdots < j_{k-1}} \eta_{j_1, \cdots, j_{k-1}}(x,t) dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}},$$

i.e., both forms $\tilde{\omega}$ and η do not contain dt. Define

$$G : \Omega^k(\mathbb{R}^n \times [0;1]) \to \Omega^{k-1}(\mathbb{R}^n)$$

by

$$G(\omega) = \sum_{j_1 < \cdots < j_{k-1}} \left(\int_0^1 \eta_{j_1, \cdots, j_{k-1}}(x,t) dt \right) dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}}.$$

Prove that

$$dG(\omega) + G(d\omega) = i_1^* \omega - i_0^* \omega.$$

(Some of you may recognize that this is a key step in proving the Poincare lemma.)

F.Q7 Sol. Since d, G, i_1^*, i_0^* are all linear operators, we versify the identify for $\omega = \tilde{\omega}$ and $\omega = dt \wedge \eta$ separately. We also only need to check for one term in the summation.

(i) $\omega = \tilde{\omega} = \tilde{\omega}_{1,2,\cdots,k}(x,t)dx^1 \wedge \cdots \wedge dx^k$ case. We compute

$$\begin{aligned} dG(\tilde{\omega}) &= d0 = 0, \\ G(d\tilde{\omega}) &= G\left(\partial_t \tilde{\omega}_{1,2,\cdots,k}(x,t)dt \wedge dx^1 \wedge \cdots \wedge dx^k\right) \\ &= \tilde{\omega}_{1,2,\cdots,k}(x,1)dx^1 \wedge \cdots \wedge dx^k - \tilde{\omega}_{1,2,\cdots,k}(x,0)dx^1 \wedge \cdots \wedge dx^k, \\ i_1^*\tilde{\omega} &= \tilde{\omega}_{1,2,\cdots,k}(x,1)dx^1 \wedge \cdots \wedge dx^k, \\ i_0^*\tilde{\omega} &= \tilde{\omega}_{1,2,\cdots,k}(x,0)dx^1 \wedge \cdots \wedge dx^k. \end{aligned}$$

The identity holds in this case.

(ii) $\omega = dt \wedge \eta = dt \wedge (\eta_{1,2,\cdots,k-1}(x,t)dx^1 \wedge \cdots \wedge dx^{k-1})$ case.

$$\begin{aligned} dG(dt \wedge \eta) &= d\left(\int_0^1 \eta_{1,\cdots,k-1}(x,t)dt \cdot dx^1 \wedge \cdots \wedge dx^{k-1}\right) \\ &= \int_0^1 \frac{\partial \eta_{1,\cdots,k-1}(x,t)}{\partial x^l} dt \cdot dx^l \wedge dx^1 \wedge \cdots \wedge dx^{k-1} \\ G(d(dt \wedge \eta)) &= G\left(\frac{\partial \eta_{1,\cdots,k-1}(x,t)}{\partial x^l} \cdot dx^l \wedge dt \wedge dx^1 \wedge \cdots \wedge dx^{k-1}\right) \\ &= -\int_0^1 \frac{\partial \eta_{1,\cdots,k-1}(x,t)}{\partial x^l} dt \cdot dx^l \wedge dx^1 \wedge \cdots \wedge dx^{k-1}, \\ i_1^*(dt \wedge \eta) &= 0 \\ i_0^*(dt \wedge \eta) &= 0. \end{aligned}$$

The identity holds in this case also. □

F.Q8. Recall the formula for Christoffel symbols Γ_{ij}^k and curvature components R_{ijkl}

$$\begin{aligned} \Gamma_{ij}^k &= \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right), \\ R_{ijkl} &= g_{lq}\left(\partial_i \Gamma_{jk}^q - \partial_j \Gamma_{ik}^q + \Gamma_{jk}^p \Gamma_{ip}^q - \Gamma_{ik}^p \Gamma_{jp}^q\right). \end{aligned}$$

(F.Q8a) Let (M^n, g) be a Riemannian manifold. Suppose (x^i) are coordinates around $p \in M$ which satisfy $g_{ij}(p) = \delta_{ij}$ and $(\partial_k g_{ij})(p) = 0$. Show that the following holds **at** p:

$$R_{ijkl} = \frac{1}{2}\left(\partial_j \partial_l g_{ik} + \partial_i \partial_k g_{jl} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il}\right).$$

(F.Q8b) Let metric $g = \frac{dx^2 + dy^2}{(1+x^2+y^2)^2}$. Compute its sectional curvature R_{1221} at point $(x,y) = (0,0)$.

F.Q8 Sol. (F.Q8a) We compute at point p

$$\begin{aligned} R_{ijkl} =& g_{lq}\left(\frac{1}{2} g^{qr} \partial_i \left(\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk}\right) - \frac{1}{2} g^{qr} \partial_j \left(\partial_i g_{kr} + \partial_k g_{ir} - \partial_r g_{ik}\right) + 0 + 0\right) \\ =& \frac{1}{2}\left(\partial_i \partial_j g_{kl} + \partial_i \partial_k g_{jl} - \partial_i \partial_l g_{jk} - \partial_j \partial_i g_{kl} - \partial_j \partial_k g_{il} + \partial_j \partial_l g_{ik}\right) \\ =& \frac{1}{2}\left(\partial_j \partial_l g_{ik} + \partial_i \partial_k g_{jl} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il}\right). \end{aligned}$$

(F.Q8b) At $(0,0)$ we have $g_{ij}) = \delta_{ij}$. By symmetry to check $\partial_k g_{ij} = 0$ at $(0,0)$ it suffice to check $\partial_x g_{11} = 0$ at $(0,0)$. This is clear from

$$\partial_x g_{11} = \partial_x (1+x^2+y^2)^{-2} = (1+x^2+y^2)^{-3} \cdot (-4x).$$

Hence at $(0,0)$

$$\begin{aligned} R_{1221} &= -2 \left(-\partial_x \frac{x}{(1+x^2+y^2)^3} - \partial_y \frac{y}{(1+x^2+y^2)^3} \right) \Bigg|_{(0,0)} \\ &= -2 \left(-\frac{1}{(1+x^2+y^2)^3} - \frac{1}{(1+x^2+y^2)^3} \right) \Bigg|_{(0,0)} \end{aligned}$$

The answer is 4. □

F.Q9. Let M^n be a closed manifold. Let ω be a closed k-form on M, $1 \le k \le n$. Prove that for any $p \in M$ there is another closed k-form τ on M which vanishes in a neighborhood of p and cohomology class $[\omega] = [\tau]$.
(Hint: Use Poincare lemma to first find a $(k-1)$-form η defined in neighborhood U of p such that $d\tilde{\eta} = \omega$ on U.

F.Q9 Sol. Choose a coordinate chart (U, φ, x) around p such that $\varphi(U) = B(0;1)$ unit ball and $\varphi(p) = 0$. Since ω is a closed form on U, by Poincare lemma there is a $(k-1)$-form η on U such that $d\eta = \omega$ on U.

Now let ρ be a smooth cutoff function on $B(0;1)$ which equals to 1 on $B(0;1/2)$. Define $\tau \doteqdot \omega - d(\rho \circ \varphi \cdot \eta)$. It is clear that $[\omega] = [\tau]$ and $\tau|_{\varphi^{-1}(B(0;1/2))} = 0$. □

F.Q10. Note that on unit sphere $(S^{n-1}, g_{S_{n-1}})$ we can attach an orthonormal moving $\bar{\omega}^i$, $i = 1, \cdots, n-1$ such that

$$d\bar{\omega}^i = \bar{\omega}^k \wedge \bar{\omega}_k^{\ i}, \qquad d\bar{\omega}_j^{\ i} - \bar{\omega}_j^{\ k} \wedge \bar{\omega}_k^{\ i} = -\bar{\omega}^j \wedge \bar{\omega}^i.$$

Consider the rotationally symmetric metric g on $(a,b) \times S^{n-1}$ defined by

$$g = dr^2 + \varphi^2(r) g_{S^{n-1}}$$

where $r \in (a,b)$ and $\varphi(r) > 0$. Use the moving frame approach to compute sectional curvatures of the metric g in the following steps.
(F.Q10a) Choose orthonormal 1-form frame ω^A using $\bar{\omega}^i$ defined above ($A = 1, 2, \cdots, n$);
(F.Q10b) Find the connection 1-forms $\omega_A^{\ B}$;
(F.Q10c) Prove that sectional curvatures

$$K_{\mathrm{rad}} = -\frac{\varphi''(r)}{\varphi(r)}, \qquad K_{\mathrm{sph}} = \frac{1 - (\varphi'(r))^2}{(\varphi(r))^2},$$

where rad stands for any plane perpendicular to hypersurface $\{r\} \times S^{n-1}$, while sph stands for any plane tangential to $\{r\} \times S^{n-1}$.

F.Q10 Sol. (F.Q10a) We choose

$$\omega^i = \varphi(r)\bar{\omega}^i, \quad \omega^n = dr.$$

(F.Q10b) To find $\omega_A^{\ B}$ we set up equations

$$\omega^n \wedge \omega_n^{\ i} + \omega^k \wedge \omega_k^{\ i} = d\omega^i = \varphi'(r)dr \wedge \bar{\omega}^i + \varphi(r)d\bar{\omega}^i = \omega^n \wedge (\varphi'(r)\bar{\omega}^i) + \omega^k \wedge \bar{\omega}_k^{\ i};$$
$$\omega^k \wedge \omega_k^{\ n} = d\omega^n = 0.$$

Some observation gives that the solution is

$$\omega_k^{\ i} = \bar{\omega}_k^{\ i}, \qquad \omega_n^{\ i} = \varphi'(r)\bar{\omega}^i.$$

(F.Q10c) We compute

$$\begin{aligned}\Omega_j^{\ i} =& d\omega_j^{\ i} - \omega_j^{\ k} \wedge \omega_k^{\ i} - \omega_j^{\ n} \wedge \omega_n^{\ i} = d\bar{\omega}_j^{\ i} - \bar{\omega}_j^{\ k} \wedge \bar{\omega}_k^{\ i} + \varphi'\bar{\omega}^j \wedge \varphi'\bar{\omega}^i \\ =& -\bar{\omega}^j \wedge \bar{\omega}^i + (\varphi')^2\bar{\omega}^j \wedge \bar{\omega}^i = -\frac{1-(\varphi')^2}{\varphi^2}\omega^j \wedge \omega^i.\end{aligned}$$

Hence sectional curvature $-K_{\text{sph}} = R_{ijji} = -\frac{1-(\varphi')^2}{\varphi^2}$.

We compute

$$\begin{aligned}\Omega_n^{\ i} =& d\omega_n^{\ i} - \omega_n^{\ k} \wedge \omega_k^{\ i} = d(\varphi'\bar{\omega}^{\ i}) - \varphi'\bar{\omega}^k \wedge \bar{\omega}_k^{\ i} \\ =& \varphi'' dr \wedge \bar{\omega}^i + \varphi'(d\bar{\omega}^i - \bar{\omega}^k \wedge \bar{\omega}_k^{\ i}) = \frac{\varphi''}{\varphi}\omega^n \wedge \omega^i.\end{aligned}$$

Hence sectional curvature $K_{\text{rad}} = R_{niin} = -\frac{\varphi''}{\varphi}$. □