Analysis Qualifying Exam - SOLUTIONS, Fall 2021

1. Let μ be a σ-finite measure on X. Let $f, g : X \to \mathbb{R}$ be measurable functions such that

$$\mu(\{x : |f(x)| > t\}) \le \mu(\{x : |g(x)| > t\}) \qquad \text{for all rational } t > 0.$$

Prove that for any $1 \le p < \infty$ we have

$$\int_X |f|^p d\mu \le \int_X |g|^p d\mu.$$

Solution: First we show that

$$\mu(\{x : |f(x)| > t\}) \le \mu(\{x : |g(x)| > t\}) \qquad \text{for all real } t > 0.$$

Indeed, let (t_n) be an increasing sequence of rational numbers such that $t_n \to t$. Then, by monotone property of measures we have

$$\mu(\{x : |f(x)| > t\}) = \lim_{n\to\infty} \mu(\{x : |f(x)| > t_n\}) \\ \le \lim_{n\to\infty} \mu(\{x : |g(x)| > t_n\}) = \mu(\{x : |g(x)| > t\}).$$

Using the distribution formula for L^p norm we have

$$\int_X |f|^p d\mu = \int_0^\infty pt^{p-1}\mu(\{x : |f(x)| > t\})dt \le \int_0^\infty pt^{p-1}\mu(\{x : |g(x)| > t\})dt = \int_X |g|^p d\mu. \quad \square$$

2. Suppose that $f : [1, \infty) \to \mathbb{C}$ is Lebesgue integrable and f is continuous at 1. Prove that

$$\lim_{n\to\infty} n \int_1^\infty \frac{f(x)}{x^n} dx = f(1).$$

Solution: Let $\delta > 1$. Observe that

$$\lim_{n\to\infty} n \int_1^\delta \frac{1}{x^n} dx = \lim_{n\to\infty} \frac{n}{-n+1} x^{-n+1}\Big|_1^\delta = 1.$$

Also observe that

$$\lim_{n\to\infty} n \left| \int_\delta^\infty \frac{f(x)}{x^n} dx \right| \le \lim_{n\to\infty} \frac{n}{\delta^n} \int_\delta^\infty |f(x)| dx = 0.$$

Therefore, it suffices to show that

$$\lim_{n\to\infty} n \int_1^\delta \frac{f(x)}{x^n} dx = \lim_{n\to\infty} n \int_1^\delta \frac{f(1)}{x^n} dx.$$

This follows from the assumption that f is continuous at 1. That is, for any $\epsilon > 0$, there exists $\delta > 1$ such that

$$|f(x) - f(1)| < \epsilon \qquad \text{for } 1 \le x \le \delta.$$

Hence,

$$\left| n \int_1^\delta \frac{f(x)}{x^n} dx - n \int_1^\delta \frac{f(1)}{x^n} dx \right| \le n \int_1^\delta \frac{|f(x) - f(1)|}{x^n} dx \le n \int_1^\delta \frac{\epsilon}{x^n} dx \to \epsilon \qquad \text{as } n \to \infty.$$

Since $\epsilon > 0$ is arbitrary, we reach the required conclusion. $\square$

3. Let (X,μ) be a positive measure space. Suppose that f and $f_1, f_2, \dots$ are functions in $L^1(X,\mu)$ such that

$$\sum_{n=1}^{\infty} \int_X |f_n - f| d\mu < \infty.$$

Show that (f_n) converges μ-almost everywhere to f.

Solution: By Chebyshev's inequality

$$\mu(\{x \in X : |f_n(x) - f(x)| > t\}) \le \frac{1}{t} \int_X |f_n - f| d\mu.$$

Hence,

$$\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x) - f(x)| > t\}) \le \frac{1}{t} \sum_{n=1}^{\infty} \int_X |f_n - f| d\mu < \infty.$$

Next we use the fact that for any sequence of measurable sets (A_n) satisfying $\sum \mu(A_n) < \infty$, almost every $x \in X$ belongs to finitely many sets A_n. We deduce that for a.e. $x \in X$,

$$|f_n(x) - f(x)| \le t \qquad \text{for all but finitely many } n \in \mathbb{N}.$$

Since $t > 0$ is arbitrary, we reach the required conclusion. □

4. Let $\mathcal{Q}$ be the collection of all dyadic intervals in $\mathbb{R}$. That is,

$$\mathcal{Q} = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) : k \in \mathbb{Z},\ n \in \mathbb{Z} \right\}.$$

Let $\mathcal{D}$ be a linear span of characteristic functions χ_Q. That is,

$$\mathcal{D} = \left\{ \sum_{Q \in \mathcal{Q}} c_Q \chi_Q : c_Q \in \mathbb{C} \text{ and } c_Q = 0 \text{ for all but finitely many } Q \in \mathcal{Q} \right\}.$$

Show that $\mathcal{D}$ is a dense subset of $L^p(\mathbb{R})$ for $1 \le p < \infty$.

Solution: We us the fact that space $C_c(\mathbb{R})$ of continuous compactly supported functions on $\mathbb{R}$ is a dense subset of $L^p(\mathbb{R})$ for $1 \le p < \infty$. Hence, it suffices to show that for any $f \in C_c(\mathbb{R})$ and $\epsilon > 0$, there exists $g \in \mathcal{D}$ such that $||f - g||_p < \epsilon$.

Since f is uniformly continuous, there exists $n > 0$ such that

$$|f(x) - f(y)| < \epsilon \qquad \text{for } |x - y| \le 2^{-n}.$$

Define

$$g(x) = \sum_{k \in \mathbb{Z}} f(k2^{-n}) \chi_{[k2^{-n},(k+1)2^{-n})}.$$

Note that supports of f and g are contained in some interval $[-R, R]$ for some $R \in \mathbb{N}$. Hence, Then,

$$\int_{\mathbb{R}} |f(x) - g(x)|^p dx = \sum_{k \in \mathbb{Z},\ [k2^{-n},(k+1)2^{-n}) \subset [-R,R]} \int_{[k2^{-n},(k+1)2^{-n})} |f(x) - f(k2^{-n})|^p dx$$
$$\le \sum_{k \in \mathbb{Z},\ [k2^{-n},(k+1)2^{-n}) \subset [-R,R]} 2^{-n} \epsilon^p dx = 2R\epsilon^p.$$

Since $\epsilon > 0$ is arbitrary, this shows that any $f \in C_c(\mathbb{R})$, and hence any function in $L^p(\mathbb{R})$, can be approximated by functions in $\mathcal{D}$ in L^p norm. □

5. Let $\{x_n\} \subset X$ be any sequence in a Banach space X. Let X^* be the space of bounded functionals on X. Prove that a sequence $\{||x_n||\}$ is bounded if and only if a sequence $\{f(x_n)\}$ is bounded for every $f \in X^*$.

Solution: One implication is trivial since for any $f \in X^*$ we have

$$|f(x_n)| \leq ||f||||x_n||.$$

Hence, we need to show that if a sequence $\{f(x_n)\}$ is bounded for every $f \in X^*$, then a sequence $\{||x_n||\}$ is bounded. Define a collection of bounded linear functionals $\{\Lambda_n\}_{n\in\mathbb{N}}$ on X^* by

$$\Lambda_n(f) = f(x_n) \qquad f \in X^*.$$

Since

$$\sup_{n\in N} |\Lambda_n(f)| < \infty \qquad \text{for all } f \in X^*,$$

by the Banach-Steinhaus theorem we have

$$\sup_{n\in N} ||\Lambda_n|| < \infty.$$

By the isometric embedding of X into X^{**}, which is given by the evaluation functional and is a consequence of the Hahn-Banach theorem, we have $||\Lambda_n|| = ||x_n||$. Hence, the required conclusion is immediate. □

6. Suppose that a vector space X has two norms $||\cdot||_1$ and $||\cdot||_2$ such that X equipped with either of them is a Banach space. Suppose that there exists a constant $C_1 > 0$ such that

$$||x||_1 \leq C_1||x||_2 \qquad \text{for all } x \in X,$$

Prove that there exists a constant $C_2 > 0$ such that

$$||x||_2 \leq C_2||x||_1 \qquad \text{for all } x \in X.$$

Solution: Consider the identity operator L between the Banach spaces $(X, ||\cdot||_1)$ and $(X, ||\cdot||_2)$. By our assumption L is a bounded linear operator. Since L is 1-to-1, the Banach invertibility theorem implies that the inverse of L, which is the identity map between the Banach spaces $(X, ||\cdot||_2)$ and $(X, ||\cdot||_1)$, is bounded as well. Hence, the required conclusion. □

7. Suppose that an entire function f satisfies

$$|f(z)| \leq \log|z| \qquad \text{for } |z| > 2021.$$

Prove that f is constant.

Solution: Let γ be a positively oriented circle $|z| = R$, where $R > 2021$. Then, by the Cauchy formula for the derivative we have

$$f'(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^2}dz \qquad \text{for } |w| < R.$$

Hence,

$$|f'(w)| \leq \frac{1}{2\pi}\sup_{|z|=R}\frac{|f(z)|}{|z-w|^2}2\pi R \leq \frac{R}{(R-|w|)^2}\log R \to 0 \qquad \text{as } R \to \infty.$$

Hence, $f'(w) = 0$ for all $w \in \mathbb{C}$. Consequently, f is constant. □

8. For a given $a \geq 0$, compute the integral

$$\int_0^\infty \frac{\cos(ax)}{1+x^2}dx.$$

Solution: Since the integrand is an even function

$$\int_{-\infty}^\infty \frac{\cos(ax)}{1+x^2}dx = 2\int_0^\infty \frac{\cos(ax)}{1+x^2}dx.$$

Consider an entire function $f(z) = \frac{e^{iaz}}{1+z^2}$. For $R > 0$, let γ_R be a path integral consisting of a line segment $[-R, R]$ and upper semi-circle C_R, which is parametrized by $t \mapsto Re^{it}$, $0 \leq t \leq \pi$. Note that

$$\int_{\gamma_R} f(z)dz = \int_{[-R,R]} \frac{\cos(ax) + i\sin(ax)}{1+x^2}dx + \int_0^\pi \frac{e^{iaR(\cos t + i\sin t)}}{1+R^2e^{2it}}Rie^{it}dt.$$

Since $x \mapsto \frac{\sin(ax)}{1+x^2}$ is an odd function

$$\int_{[-R,R]} \frac{\cos(ax)+i\sin(ax)}{1+x^2}dx = \int_{[-R,R]} \frac{\cos(ax)}{1+x^2}dx \to \int_{-\infty}^\infty \frac{\cos(ax)}{1+x^2}dx \qquad \text{as } R \to \infty.$$

The second part of the path integral is dominated by

$$\int_0^\pi \frac{|e^{iaR(\cos t+i\sin t)}|}{|1+R^2e^{2it}|}Rdt = \int_0^\pi \frac{e^{-aR\sin t}}{|1+R^2e^{2it}|}Rdt \leq \frac{\pi R}{R^2-1} \to 0 \qquad \text{as } R \to \infty.$$

Since f has poles at $z = \pi i$, by the residue theorem

$$\int_{\gamma_R} f(z)dz = 2\pi i \operatorname{Res}(f, i) = 2\pi i \operatorname{Res}\left(\frac{e^{iaz}}{(z+i)(z-i)}, i\right) = \pi e^{-a}.$$

Consequently,

$$\int_0^\infty \frac{\cos(ax)}{1+x^2}dx = \frac{1}{2}\pi e^{-a}. \quad \square$$

9. Let $\Omega \subset \mathbb{C}$ be open and bounded. Let $C(\overline{\Omega})$ be the space of continuous functions on the closure of Ω. Show that

$$\{f \in C(\overline{\Omega}) : f \text{ is holomorphic on } \Omega\}$$

is a closed subspace of $C(\overline{\Omega})$.

Solution: Suppose that $\{f_n\}$ is a sequence of holomorphic functions on Ω, which converges uniformly to some function $f \in C(\overline{\Omega})$. It suffices to show that f is holomorphic as well. This is a consequence of the Morera's theorem. Indeed, for any $z \in \Omega$, choose a disk $D(z,r)$ such that $\overline{D(z,r)} \subset \Omega$. For any triangle $\Delta \subset D(z,r)$ the path integral over its edges

$$\int_{\partial\Delta} f_n(z)dz = 0.$$

Since f_n converges uniformly to f we deduce that

$$\int_{\partial\Delta} f(z)dz = 0.$$

By Morera's theorem, this implies that f is holomorphic on $D(z,r)$. Since $z \in \Omega$ is arbitrary, we are done. $\square$