Algebra Qualifying Exam **Fall 2021**

Part I. True/False questions (9 points each). Give brief but to the point justification.

1. Every torsion-free $\mathbb{C}[x]$-module is free.
2. Every short exact sequence of abelian groups $0 \to \mathbb{Q}/\mathbb{Z} \to A \to B \to 0$ splits.
3. Let $F_1, F_2 : A - \text{mod} \to B - \text{mod}$ be functors of the form $F_i(M) = P_i \otimes_A M$, $i = 1, 2$, where P_i are $B - A$-bimodules. If F_1 and F_2 are isomorphic as functors then P_1 and P_2 are isomorphic as $B - A$-bimodules.
4. Every Galois extension E/K of degree 18 contains a Galois subextension L/K of degree 9.
5. If R is a noetherian commutative ring and $J(R)$ is its Jacobson radical, then every $R/J(R)$-module is completely reducible.

II. Longer problems (12 points each). Do any *four* of the following problems.

1. Let A be a finite-dimensional semisimple algebra over a field F, and let $Z(A) \subset A$ denote its center. Let M_1 and M_2 be simple A-modules. Prove that if M_1 and M_2 are isomorphic as $Z(A)$-modules then they are isomorphic as A-modules.
2. Let $K \subset \mathbb{C}[x, y, z]$ denote the kernel of the homomorphism

   $$f : \mathbb{C}[x, y, z] \to \mathbb{C}[t] : x \mapsto t^2, \quad y \mapsto t^3, \quad z \mapsto t^4.$$

   (a) Prove that K is a prime ideal. (b) Prove that any prime ideal $I \subset \mathbb{C}[x, y, z]$ such that $K \subset I$ and $K \neq I$, is maximal.
3. Here is a partially completed character table of a finite group (where c_i is the cardinality of the conjugacy class C_i).

C_i	C_1	C_2	C_3	C_4	C_5
c_i	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	
χ_4	1	1	-1	-1	
χ_5					

   Complete the table. Determine the isomorphism types of the groups $Z(G)$ and G/G'.
4. Classify up to similarity all linear tansformations $T \in \text{End}_{\mathbb{C}^6}(\mathbb{C}^6)$ such that $T^6 = 0$ and T has at most two 2-dimensional invariant subspaces.
5. Let G be a group of order $2^3 \cdot 7^2 \cdot 11$. Show that G has a subgroup of order 77.

Solutions.

I.1. False. The $\mathbb{C}[x]$-module $\mathbb{C}(x)$ is not free. Any two elements of $\mathbb{C}(x)$ are linearly dependent over $\mathbb{C}[x]$, so a basis can consists only of 1 element, but $\mathbb{C}(x)$ is not generated by 1 element over $\mathbb{C}[x]$.

I.2. True. The module $\mathbb{Q}/\mathbb{Z}$ is divisible, hence injective.

I.3. True. An isomorphism of functors $F_1 \to F_2$ gives an isomorphism of B-modules $P_1 \simeq F_1(A) \to F_2(A) \simeq P_2$, which commutes with endomorphisms induced by right multiplication of A on itself, i.e., with the right A-module structures. Hence, we get an isomorphism of $B-A$-bimodules.

I.4. False. Any finite group G can be realized as a Galois group. Now take G to be the dihedral group of order 18. It does not have a normal subgroup of order 2.

I.5. False. Take $R = \mathbb{C}[x]$. Then $J(\mathbb{C}[x]) = 0$ but the module $\mathbb{C}[x]/(x^2)$ is not completely reducible.

II.1. Consider the decomposition $A = M_{n_1}(D_1) \oplus \ldots \oplus M_{n_r}(D_r)$, where D_i are division algebras over F. Then the center of A is $Z(A) = Z(D_1) \oplus \ldots \oplus Z(D_r)$. Let $e_1, \ldots, e_r$ denote the central idempotents in A. A simple A-module M is isomorphic to $D_i^{n_i}$, so its isomorphism class is determined by the unique i such that $e_i M = M$ and $e_j M = 0$ for $j \neq i$. Since e_i belong to the center $Z(A)$, we can recover the isomorphism class of M by viewing M as a $Z(A)$-module.

II.2. The image of f is the subring $\mathbb{C}[t^2, t^3] \subset \mathbb{C}[t]$. In particular, it is a domain, so K is prime. Prime ideals of $\mathbb{C}[x, y, z]$ properly containing K correspond to nonzero prime ideals in $\mathrm{im}(f) \simeq \mathbb{C}[t^2, t^3]$. Since $\mathbb{C}[t^2, t^3] \subset \mathbb{C}[t]$ is an integral extension, we have $\dim \mathbb{C}[t^2, t^3] = \dim \mathbb{C}[t] = 1$, so any nonzero prime ideal in $\mathbb{C}[t^2, t^3]$ is maximal.

II.3. One completes the table using orthogonalities:

C_i	C_1	C_2	C_3	C_4	C_5
c_i	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

We have $Z(G) = C_1 \cup C_2$, so $Z(G) \simeq \mathbb{Z}_2$. $|G/G'|$ is the number of 1-dimensional representations, so G/G' is an abelian group of order 4. Since the 4th root of unity does not appear in the character table, we have $G/G' \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

II.4. T can have at most 2 Jordan blocks (with eigenvalues 0): otherwise we get more than two 2-dimensional invariant subspaces. If T is a single Jordan block, then the corresponding $\mathbb{C}[x]$-module, $\mathbb{C}[x]/(x^6)$ has a unique 2-dimensional invariant subspace. If T has 2 Jordan blocks, both of size ≥ 2 then one can still construct three 2-dimensional invariant subspaces. Finally, for the $\mathbb{C}[x]$-module $M = \mathbb{C}[x]/(x) \oplus \mathbb{C}[x]/(x^5)$ we can also construct three distinct 2-dimensional submodules: $M_1 = \mathbb{C}[x]/(x) + (x^4)/(x^5)$, $M_2 = 0 \oplus (x^3)/(x^5)$ and $M_3 = \mathbb{C}[x] \cdot (1, x^3)$. Thus, only the single Jordan block works.

II.5 The number n_{11} of Sylow 11-subgroups of G divides $2^3 \cdot 7^2$ and is $\equiv 1 \bmod (11)$. This is possible only if either $n_{11} = 1$ or $n_{11} = 56$. In the latter case, $|N(P_{11})| = 77$, so we are done. In the former case, P_{11} is normal and we can take the preimage of a subgroup of order 7 in G/P_{11}.