Probability Qualifying Exam 2021

There are eight problems on this test. Read each problem carefully before beginning. PARTIAL CREDIT CANNOT BE AWARDED UNLESS YOUR WORK IS CLEAR.

Problem	Possible Points	Earned Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

Problem 1. Let $a \wedge b := \min\{a, b\}$.

For random variables X, Y (defined on the same probability space), let

$$\rho(X, Y) := \mathbb{E}(|X - Y| \wedge 1)\,.$$

For sequences $\{(X_n, Y_n)\}$, where X_n and Y_n are defined on the same probability space, show that $\lim_{n\to\infty} \rho(X_n, Y_n) = 0$ if and only if $X_n - Y_n \to 0$ **in probability**.

Problem 2. Suppose that $X_1, X_2, \dots$ are random variables satisfying $\mathbb{E}(X_i) = 0$ and such that $\mathbb{E}(X_i^2) < \infty$ for all i. Assume further that there is a constant $C_1 > 0$ (not depending on i or j) so that

$$\mathbb{E}[X_i X_j] \leq \frac{C_1}{|i-j|^2 + 1}.$$

Let $S_n = \sum_{i=1}^n X_i$.

(a) Show that $\mathrm{Var}(S_{n+m} - S_n) \leq C_2 m$ for all n, m, where C_2 is a constant (not depending on n, m).

(b) Use (a) to show that $\lim_{n\to\infty} \frac{S_{n^2}}{n^2} = 0$ a.s.

(c) Show that $\lim_{n\to\infty} \frac{S_n}{n} = 0$ a.s.
Hint: Show that the events $A_n = \{\bigcup_{m \leq 2n} |S_{n^2+m} - S_{n^2}| > \varepsilon n^2\}$ occur only finitely often.

Problem 3. Show that if $\{X_{n,i}\}$ are i.i.d. with $X_{n,i} \in \{0,1\}$ and $\mathbb{E}(X_{n,i}) = p_n$, and $np_i \to \lambda$, then $S_n = \sum_{i=1}^{n} X_{n,i}$ converges in distribution to a Poisson(λ) distribution.

Problem 4. Let $\{Z_{n,k}\}$ be i.i.d. non-negative, integer-valued random variables with $\mathbb{E}(Z_{n,k}) = \mu$. Define, recursively,

$$X_n = \sum_{k=1}^{X_{n-1}} Z_{n,k}\,.$$

Note $\{X_n\}$ is a branching process with average number of offspring per individual equal to μ.

Show that $\frac{X_n}{\mu^n}$ converges to a finite limit (possibly random) as $n \to \infty$ almost surely.

Problem 5. Suppose that $\{X_n\}$ is the Markov chain on $\{0, 1, \ldots\}$ which satisfies, for $k \geq 1$,

$$P(k, k+1) = 1 - P(k, k-1) = p < \frac{1}{2},$$

and $P(0,0) = 1 - p = 1 - P(0,1)$. (Informally, $\{X_n\}$ is a nearest-neighbor walk on $\mathbb{N}$ which moves up with probability p and down with probability $1-p$.)

Show that $\mathbb{E}_0(\tau_0) < \infty$, where

$$\tau_0 = \min\{n \geq 1 : X_n = 0\}.$$

Problem 6. Let $(B_t)_{t\geq 0}$ be standard Brownian motion with $B_0 = 0$. Let

$$\tau := \inf\{t \geq 0 : B_t \notin (a, b)\}, \qquad \text{for } a < 0 < b.$$

Let

$$M_t = B_t^3 - 3\int_0^t B_u du.$$

Show that $(M_t)_{t\geq 0}$ is a martingale, and find an expression (in terms of a, b only) for $\mathbb{E}\left[\int_0^\tau B_u du\right]$.

You may find useful that $\mathbb{P}(B_\tau = a) = b/(b-a)$, and may use the fact that $\mathbb{E}[\tau] < \infty$ without proof. State carefully any optional stopping theorem you apply, and show the conditions are satisfied.

Problem 7. Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain on an infinite state space S with transition matrix P, and for an arbitrary subset F of S let $\tau_F = \inf\{t \geq 0 : X_n \in F\}$ be the first hitting time of F. Fix **finite** disjoint subsets A and B of S and let $\mathscr{H}$ be the set of functions $h : S \to \mathbb{R}$ that solve the following system of equations:

$$\begin{aligned} Ph(x) &= h(x) \qquad \text{for } x \notin A \cup B \\ h(x) &= 1 \qquad \text{for } x \in A \\ h(x) &= 0 \qquad \text{for } x \in B. \end{aligned}$$

(a) Show that $h_{A,B}(x) = \mathbb{P}_x\{\tau_A < \tau_B\}$ is in $\mathscr{H}$.

(b) Show that if the chain is transient, then $h_{A,B}$ is *not* the only member of $\mathscr{H}$ by explicitly constructing other solutions.

Hint: consider the function $e(x) = \mathbb{P}_x\{\tau_{A\cup B} = \infty\}$.

Problem 8. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, and define $W_t = tB_{1/t}$. Show that $(W_t)_{t\geq 0}$ is also a standard Brownian motion.