Qualifying Exam, Differential Geometry
Fall 2021

F.Q1. Let S^2 denote the unit sphere in $\mathbb{R}^3$ which has standard coordinates (x, y, z). Show that the map

$$f : S^2 \to \mathbb{R}^3, \qquad f(x, y, z) = ((1 - z^2)x, (1 - z^2)y, z)$$

is a smooth one-to-one map but not a smooth embedding.

F.Q2. Let g be a **continuous** Riemannian metric tensor on a smooth manifold M^n (not-necessarily-compact). Show that there is a **smooth** Riemannian metric tensor h on M such that bilinear map

$$f(p) - h(p) : T_pM \times T_pM \to \mathbb{R}$$

is positive definite for all $p \in M$.

F.Q3. Construct an explicit diffeomorphism from $\mathbb{R}^3 \setminus \bar{E}$ to $S^2 \times \mathbb{R}$. Here S^2 is round sphere of radius 1 and $\bar{E}$ is closed ellipsoid

$$\bar{E} = \{(x, y, z) \in \mathbb{R}^3, \ \frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{5^2} \leq 1\}.$$

F.Q4. A smooth real-valued function f, defined on some open subset $U \subset \mathbb{R}^n$, is called **harmonic** if $\sum_{i=1}^n \frac{\partial^2 f}{(\partial x^i)^2} = 0$ on U. Show that f is harmonic if and only if for every $p \in U$ and every positive number r with closed ball $\bar{B}(p; r) \subset U$ of radius r and center p, we have

$$\sum_{i=1}^n (-1)^i \int_{S(p;r)} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n = 0,$$

where $S(p; r) = \partial \bar{B}(p; r)$ is the boundary, and $\widehat{dx^i}$ indicates that dx^i is omitted from the wedge product.

F.Q5. Let M^n be an oriented closed manifold and let $U \subset M$ be an open set. Fix a smooth n-form ω on M with $\int_M \omega = 1$.

(F.Q5a) Prove that there is a smooth n-form ω_1 with support $\operatorname{supp} \omega_1 \subset U$ such that $\omega - \omega_1 = d\eta$ for some $(n-1)$-form η on M.

(F.Q5b) Let N^n be a closed manifold with an orientation and let $F : N \to M$ be a smooth map such that $q \in M$ is a regular value with $F^{-1}(q) = \{p_1, \cdots, p_k\}$. Porve that $|\int_N F^*\omega| \leq k$.

F.Q6. If $\eta = \eta_i dx^i$ is a 1-form on some Riemannian manifold (M^n, g), let $\eta_{i,jk} dx^i \otimes dx^j \otimes dx^k$ be the local expression for $\nabla^2 \eta$. Prove the Ricci identity

$$\eta_{i,jk} - \eta_{i,kj} = R_{jki}{}^l \eta_l.$$

F.Q7. Let $\pi : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ denote the standard projection. For each $t \in [0,1]$, consider the map

$$i_t : \mathbb{R}^n \to \mathbb{R}^n \times [0,1], \qquad i_t(x) = (x,t),$$

where $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$. If ω is a smooth k-form in $\Omega^k(\mathbb{R}^n \times [0,1])$, then obviously ω can be written uniquely as

$$\omega = \tilde{\omega} + dt \wedge \eta$$

where

$$\tilde{\omega} = \sum_{i_1 < \cdots < i_k} \tilde{\omega}_{i_1, \cdots, i_k}(x,t) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and

$$\eta = \sum_{j_1 < \cdots < j_{k-1}} \eta_{j_1, \cdots, j_{k-1}}(x,t) dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}},$$

i.e., both forms $\tilde{\omega}$ and η do not contain dt. Define

$$G : \Omega^k(\mathbb{R}^n \times [0;1]) \to \Omega^{k-1}(\mathbb{R}^n)$$

by

$$G(\omega) = \sum_{j_1 < \cdots < j_{k-1}} \left(\int_0^1 \eta_{j_1, \cdots, j_{k-1}}(x,t) dt \right) dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}}.$$

Prove that

$$dG(\omega) + G(d\omega) = i_1^* \omega - i_0^* \omega.$$

(Some of you may recognize that this is a key step in proving the Poincare lemma.)

F.Q8. Recall the formula for Christoffel symbols Γ_{ij}^k and curvature components R_{ijkl}

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right),$$
$$R_{ijkl} = g_{lq} \left(\partial_i \Gamma_{jk}^q - \partial_j \Gamma_{ik}^q + \Gamma_{jk}^p \Gamma_{ip}^q - \Gamma_{ik}^p \Gamma_{jp}^q \right).$$

(F.Q8a) Let (M^n, g) be a Riemannian manifold. Suppose (x^i) are coordinates around $p \in M$ which satisfy $g_{ij}(p) = \delta_{ij}$ and $(\partial_k g_{ij})(p) = 0$. Show that the following holds **at** p:

$$R_{ijkl} = \frac{1}{2} \left(\partial_j \partial_l g_{ik} + \partial_i \partial_k g_{jl} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il} \right).$$

(F.Q8b) Let metric $g = \frac{dx^2 + dy^2}{(1+x^2+y^2)^2}$. Compute its sectional curvature R_{1221} at point $(x,y) = (0,0)$.

F.Q9. Let M^n be a closed manifold. Let ω be a closed k-form on M, $1 \leq k \leq n$. Prove that for any $p \in M$ there is another closed k-form τ on M which vanishes in a neighborhood of p and cohomology class $[\omega] = [\tau]$.
(Hint: Use Poincare lemma to first find a $(k-1)$-form η defined in neighborhood U of p such that $d\tilde{\eta} = \omega$ on U.)

F.Q10. Note that on unit sphere $(S^{n-1}, g_{S_{n-1}})$ we can attach an orthonormal moving $\bar{\omega}^i$, $i = 1, \cdots, n-1$ such that

$$d\bar{\omega}^i = \bar{\omega}^k \wedge \bar{\omega}_k{}^i, \qquad d\bar{\omega}_j{}^i - \bar{\omega}_j{}^k \wedge \bar{\omega}_k{}^i = -\bar{\omega}^j \wedge \bar{\omega}^i.$$

Consider the rotationally symmetric metric g on $(a, b) \times S^{n-1}$ defined by

$$g = dr^2 + \varphi^2(r) g_{S^{n-1}}$$

where $r \in (a, b)$ and $\varphi(r) > 0$. Use the moving frame approach to compute sectional curvatures of the metric g in the following steps.

(F.Q10a) Choose orthonormal 1-form frame ω^A using $\bar{\omega}^i$ defined above $(A = 1, 2, \cdots, n)$;

(F.Q10b) Find the connection 1-forms $\omega_A{}^B$;

(F.Q10c) Prove that sectional curvatures

$$K_{\mathrm{rad}} = -\frac{\varphi''(r)}{\varphi(r)}, \qquad K_{\mathrm{sph}} = \frac{1 - (\varphi'(r))^2}{(\varphi(r))^2},$$

where rad stands for any plane perpendicular to hypersurface $\{r\} \times S^{n-1}$, while sph stands for any plane tangential to $\{r\} \times S^{n-1}$.