Analysis Qualifying Exam, Fall 2021

1. Let μ be a σ-finite measure on X. Let $f, g : X \to \mathbb{R}$ be measurable functions such that

$$\mu(\{x : |f(x)| > t\}) \leq \mu(\{x : |g(x)| > t\}) \qquad \text{for all rational } t > 0.$$

Prove that for any $1 \leq p < \infty$ we have

$$\int_X |f|^p d\mu \leq \int_X |g|^p d\mu.$$

2. Suppose that $f : [1, \infty) \to \mathbb{C}$ is Lebesgue integrable and f is continuous at 1. Prove that

$$\lim_{n\to\infty} n \int_1^\infty \frac{f(x)}{x^n} dx = f(1).$$

3. Let (X, μ) be a positive measure space. Suppose that f and $f_1, f_2, \ldots$ are functions in $L^1(X, \mu)$ such that

$$\sum_{n=1}^\infty \int_X |f_n - f| d\mu < \infty.$$

Show that (f_n) converges μ-almost everywhere to f.

4. Let $\mathcal{Q}$ be the collection of all dyadic intervals in $\mathbb{R}$. That is,

$$\mathcal{Q} = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) : k \in \mathbb{Z},\ n \in \mathbb{Z} \right\}.$$

Let $\mathcal{D}$ be a linear span of characteristic functions χ_Q. That is,

$$\mathcal{D} = \left\{ \sum_{Q \in \mathcal{Q}} c_Q \chi_Q : c_Q \in \mathbb{C} \text{ and } c_Q = 0 \text{ for all but finitely many } Q \in \mathcal{Q} \right\}.$$

Show that $\mathcal{D}$ is a dense subset of $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

5. Let $\{x_n\} \subset X$ be any sequence in a Banach space X. Let X^* be the space of bounded functionals on X. Prove that a sequence $\{||x_n||\}$ is bounded if and only if a sequence $\{f(x_n)\}$ is bounded for every $f \in X^*$.

6. Suppose that a vector space X has two norms $|| \cdot ||_1$ and $|| \cdot ||_2$ such that X equipped with either of them is a Banach space. Suppose that there exists a constant $C_1 > 0$ such that

$$||x||_1 \leq C_1 ||x||_2 \qquad \text{for all } x \in X,$$

Prove that there exists a constant $C_2 > 0$ such that

$$||x||_2 \leq C_2 ||x||_1 \qquad \text{for all } x \in X.$$

7. Suppose that an entire function f satisfies

$$|f(z)| \leq \log |z| \qquad \text{for } |z| > 2021.$$

Prove that f is constant.

8. For a given $a \geq 0$, compute the integral

$$\int_0^\infty \frac{\cos(ax)}{1 + x^2} dx.$$

9. Let $\Omega \subset \mathbb{C}$ be open and bounded. Let $C(\overline{\Omega})$ be the space of continuous functions on the closure of Ω. Show that

$$\{f \in C(\overline{\Omega}) : f \text{ is holomorphic on } \Omega\}$$

is a closed subspace of $C(\overline{\Omega})$.