Algebra Qualifying Exam **Fall 2021**

Part I. True/False questions (9 points each). Give brief but to the point justification.

1. Every torsion-free $\mathbb{C}[x]$-module is free.

2. Every short exact sequence of abelian groups $0 \to \mathbb{Q}/\mathbb{Z} \to A \to B \to 0$ splits.

3. Let $F_1, F_2 : A - \text{mod} \to B - \text{mod}$ be functors of the form $F_i(M) = P_i \otimes_A M$, $i = 1, 2$, where P_i are $B - A$-bimodules. If F_1 and F_2 are isomorphic as functors then P_1 and P_2 are isomorphic as $B - A$-bimodules.

4. Every Galois extension E/K of degree 18 contains a Galois subextension L/K of degree 9.

5. If R is a noetherian commutative ring and $J(R)$ is its Jacobson radical, then every $R/J(R)$-module is completely reducible.

II. Longer problems (12 points each). Do any *four* of the following problems.

1. Let A be a finite-dimensional semisimple algebra over a field F, and let $Z(A) \subset A$ denote its center. Let M_1 and M_2 be simple A-modules. Prove that if M_1 and M_2 are isomorphic as $Z(A)$-modules then they are isomorphic as A-modules.

2. Let $K \subset \mathbb{C}[x, y, z]$ denote the kernel of the homomorphism

$$f : \mathbb{C}[x, y, z] \to \mathbb{C}[t] : x \mapsto t^2, \quad y \mapsto t^3, \quad z \mapsto t^4.$$

   (a) Prove that K is a prime ideal. (b) Prove that any prime ideal $I \subset \mathbb{C}[x, y, z]$ such that $K \subset I$ and $K \neq I$, is maximal.

3. Here is a partially completed character table of a finite group (where c_i is the cardinality of the conjugacy class C_i).

C_i	C_1	C_2	C_3	C_4	C_5
c_i	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	
χ_4	1	1	-1	-1	
χ_5					

   Complete the table. Determine the isomorphism types of the groups $Z(G)$ and G/G'.

4. Classify up to similarity all linear tansformations $T \in \text{End}_{\mathbb{C}^6}(\mathbb{C}^6)$ such that $T^6 = 0$ and T has at most two 2-dimensional invariant subspaces.

5. Let G be a group of order $2^3 \cdot 7^2 \cdot 11$. Show that G has a subgroup of order 77.