MATH 253: Calculus III
Model Syllabus 2021-2022
Course coordinator: Dan Dugger

Instructor: Put your name and contact information and office hours.

Prerequisite: C- or better in Math 112, or satisfactory placement exam score.

Text: *OpenSTAX Calculus Volume I.* An electronic edition of this text is available for free at

http://openstax.org/details/books/calculus-volume-1

The course covers most of Chapters 5–6.

> Note: For the past couple of years we have been in the process of transitioning from Stewart to OpenSTAX. As of this year we would like to complete the move and have everyone use OpenSTAX. However, if there is a compelling reason for an instructor to stick with Stewart you can bring the issue to the course coordinator, Director of Undergraduate Studies, or Department Head for consideration.
> If you find errors in OpenSTAX, or things you wish were done differently, please keep track of these and send a list to the course coordinator at some point. We plan to repair these errors and improve the text over time.

Overview: The course should cover Chapters 5–6 together with power series solutions to differential equations (some of this is in Section 6.4, and that might be enough).

Chapters 5–6 should be thought of as building to Taylor Polynomials and Taylor's Remainder Theorem, and also to power series representation of functions. Along the way, one considers sequences and sequence convergence, series, and series convergence.

Finally, as an additional application of power series, we'd like to cover power series solutions to differential equations. Some of this in in Section 6.4, and that might be all you want to cover. This material should only be covered in the most elementary way considering differential equations of the form

$$y' + p(x)y = q(x) \text{ and } y'' + p(x)y' + q(x)y = r(x)$$

for p, q, r polynomials, and thus staying away from discussions of things like ordinary points, singular points, regular singular points, etc. Chapter 12.8 from Marsden and Weinstein's text *Calculus II* allegedly does a good job with this subject, and it is available for free here:

http://authors.library.caltech.edu/25036/2/Calc2w.pdf

Exams: I've written a schedule for two midterms and a final. One midterm is really not a good idea, since the students in this course need more feedback rather than less.

You should put the time of your final exam, from the registrar's website based on your class starting time, on the syllabus.

Grade Scheme:

Homework	20%
Midterm 1	25%
Midterm 2	25%
Final Exam	30%

Instructors should feel free to change this system a bit as they see fit, but the above is fairly typical. Large changes should be discussed with the course coordinator.

Homework: It is hard to use WebWork as the primary homework tool in this class, as most problems involve deciding whether or not a certain sequence/series converges and justifying the answer. A few problems on WebWork might be used for supplementary purposes, but it's not clear if this is worth it. See

`http://pages.uoregon.edu/ddugger/ma253.html`

for the homework assignments I last used—those were from Stewart, but the problems could be copied and used in a course based on OpenSTAX. It is not clear that the OpenSTAX book has enough good problems on its own.

Be warned that if you assign problems from a book then many students will find and copy solutions from the internet. It can be a good idea to give the problems on a worksheet without any reference to a textbook, to add an extra barrier to this kind of thing. But there are also websites such as Chegg where students can get problems solved for them. Be aware that cheating on homework is very easy these days.

Workload: There will be homework due every week, as well as reading and class attendance. Some years I have broken up the homework assignment and had the problems due twice a week, say on Tuesdays and Fridays—this keeps students from putting everything off until the last minute and not practicting the skills that are being used in lecture.

An average well-prepared student should expect to spend about 12 hours per week on this course (including time in class), but there will be a lot of variation depending on background and ability.

Broad Course Learning Goals:

The primary goal of the course is to bring students to a point where they can ***use* Taylor's theorem in a reasonably effective way**; at least on standard Taylor polynomial approximations like those for $\sin(x)$, $\cos(x)$, e^x and $\log(x)$.

This means they need to be able to compute the Taylor polynomials, and then (this is the difficult part) *use Taylor's theorem to estimate the error!* The remainder theorem appears in 6.3, and applications of the remainder theorem are section 6.4. Note that this comes well into the term, and you want be sure and reach this point when there are enough weeks left in the term to give students practice doing the sorts of exercises that occur in 6.4 before the final.

Here is a list of course goals that includes some less central points

- Show sequences *don't* converge by using ϵ-N definition of limit.
- Use standard series convergence tests.
- Estimate sums using the integral test when possible, the alternating series test when possible, and the comparison test when possible.
- Calculate radii of convergence for a power series, calculate Taylor series, represent common transcendental functions as power series.
- **Use Taylor's remainder theorem to approximate values of transcendental functions to given levels of accuracy.**
- Give power series solutions to appropriate differential equations. Recognize solutions when they are common transcendental functions.

Note that the course goals above emphasize applications and this is what the course should do in general. But it is appropriate to introduce the precise definition of limit of a sequence and explain that one needs this definition to prove various facts that will be stated without proof in the course. In addition, one could then use that definition to prove a very small number of elementary things. For example, one could prove that a given sequence can't have two different limits, and one could use the definition to prove that certain sequences don't have limits.

More Detailed Learning Goals: All sections of 253 should cover learning goals (1)–(18) below. Some instructors may wish to cover (19) as well. If you are adopting additional learning goals, that should be discussed in advance with the course coordinator.

(1) Decide if a given sequence converges or not.

(2) Express an indicated sum using Σ notation in closed form.

(3) Compute partial sums and other finite sums.

(4) State the precise definition of what it means for a sequence to have a limit.

(5) State the precise definition of what it means for a series to converge.

(6) Decide if a given series converges or not, using the Comparison Test, Divergence Test, Root Test, Ratio Test, Integral Test, Limit Comparison Test, Alternating Series Test, or a combination thereof, as appropriate.

(7) Decide if a given series converges or not using the definition.

(8) Evaluate the Taylor polynomial for a given function, given a center and a degree, by computing derivatives.

(9) Compute the Taylor polynomial for a rational function by performing long division.

(10) Use Taylor polynomials to approximate the values of functions.

(11) Given an easy sequence that converges to a limit L, together with an ϵ, determine an N such that $|a_n - L| < \epsilon$ for $n \geq N$.

(12) Given an alternating series and an epsilon, determine how many terms are needed to have the partial sum within epsilon of the limit.

(13) Find the interval of convergence of a given power series.

(14) Determine if a given series is absolutely convergent.

(15) Given a function, a center a, a degree d, and an accuracy level ϵ, determine an interval about a for which the dth Taylor polynomial is within ϵ of the function at all points.

(16) Use Taylor's Inequality to bound the error of a Taylor approximation.

(17) Given a differential equation, find the general solution as a power series up through a given degree. Also find particular solutions.

(18) Answer basic conceptual questions involving convergence of sequences and series, and also give examples of related phenomena.

Optional learning outcomes:

(19) Use and apply modern technology (e.g., computer software) in some way that engages with the other learning outcomes.

Warnings: This is a difficult class to teach. The applications of the material to science students are not as readily accessible as they were in 251-252, and the problems are not just about "getting an answer" anymore. Understanding the subtleties of convergence requires a logical and mathematical framework that most students are encountering for the first time, and they need a lot of help with this. It possibly makes sense to develop more training material along these lines, to be included in the course, but we are not there yet.

Here are some specific comments:

(1) You cannot assume that students know what factorials are, or what binomial coefficients are. Make sure to spend adequate time explaining these things.
(2) Students have no idea what is bigger than what. It is useful to spend some time going over why

$$\text{constant} < \log x < x^{0.1} < x < x^{1.1} < x^2 < 2^x < 3^x < x! < x^x$$

(where for some of these inequalities I am being sloppy about the applicable values of x).
(3) Learning precise mathematical statements of definitions and theorems is a key part of this course. Don't expect students to realize that they should do this, or even to know *how* to do it. It is best to be clear about your expectations here, and to give the students practice with this before assessing them on an exam.
(4) There's a certain sense to starting the class with Taylor polynomials and with the whole idea of approximating a function by a polynomial. You can even talk about approximate solutions of ODEs in a naive sense. At some point this naturally leads into the idea of infinite series, and that brings you to the convergence material in a way that highly motivates it. Note that this approach allows students to work with Taylor's formula for the whole course, rather than just restricting it to the last half. Unfortunately, the textbook is not written this way... I have taught the course in this manner and felt like it worked well, though. See the second schedule below.

Technology: This is a good course to incorporate modern technology, if you feel so inclined. It is useful for students to learn how to use Mathematica to crank out Taylor approximations, compute finite sums of series, and other tedious tasks. Mathematica can also be a good tool for exploring questions around convergence and around Taylor's Theorem. For some homework questions devoted to this kind of thing, see my webpage from the last time I taught the course (cited in the Homework section above). What we have now is only a beginning, and it would be nice to develop homework problems that use the technology in more interesting ways. This stuff is also available (with TeX files) in the department's "calculus materials depository".

Learning Environment: The University of Oregon strives for inclusive learning environments. Please notify me if the instruction or design of this course results in disability-related barriers to your participation. You are also encouraged to contact the Accessible Education Center in 360 Oregon Hall at 541-346-1155 or uoaec@uoregon.edu.

Academic Conduct: The code of student conduct and community standards is at conduct.uoregon.edu. In this course, it is appropriate to help each other on homework as long as the work you are submitting is your own and you understand it. It is not appropriate to help each other on exams, to look at other students' exams, or to bring unauthorized material to exams.

Approximate Schedule (classical approach)

Week 1	5.1	Sequences
Week 2	5.2, 5.3	Series, Divergence and Integral Tests
Week 3	5.3, 5.4	Divergence and Integral Tests, Comparison tests
Week 4	5.5, 5.6	Alternating series, ratio and root tests
Week 5	6.1, 6.2	Power series
Week 6	6.3	Taylor and Maclaurin series
Week 7	6.3	More on Taylor series
Week 8	6.4	Applications of Taylor series
Week 9	6.4	Power series solutions to diff eqs
Week 10		Review, catch-up

When I teach this course I like to start with Taylor polynomials and Taylor approximations, as an extension of the idea of linear approximation from Math 251: find polynomials that match the derivatives of a given function up through some order. This quickly leads to the idea of an infinite series, thereby motivating what we will spend most of the course on. As an extra benefit, getting students working with Taylor polynomials from Day 1 lets me write homework assignments that include computational problems along with the more theoretical convergence test problems that overtake the course in the first few weeks. So here is an alternative schedule incorporating this idea:

Approximate Schedule (alternative approach)

Week 1	6.3	Taylor polynomials as approximations
Week 2	5.1	Sequences
Week 3	5.2, 5.3	Series, Divergence and Integral Tests
Week 4	5.3, 5.4	Divergence and Integral Tests, Comparison tests
Week 5	5.5, 5.6	Alternating series, ratio and root tests
Week 6	6.1, 6.2	Power series
Week 7	6.3	Taylor and Maclaurin series
Week 8	6.4	Applications of Taylor series
Week 9	6.4	Power series solutions to diff eqs
Week 10		Review, catch-up