1	2	3	4	5	6	7	8	9	10	Total

QUALIFYING EXAM, Winter 2021

Algebraic Topology

NAME ____

(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ____________________ **SIGNATURE** ____________________

Do all 10 problems. Please write clearly.

Problem 1 State and prove the Jordan-Brouwer Theorem. (If you would like to use some preliminary results, please state them clearly.)

Problem 2 Define the Hopf invariant $h : \pi_{2k-1}S^k \to \mathbf{Z}$. Assume the Hopf invariant is a homomorphism. Let $\iota_{2n} : S^{2n} \to S^{2n}$ be the identity map. Prove that $h([\iota_{2n}, \iota_{2n}])$ is non-zero, and use this to prove that $\pi_{4n-1}(S^{2n})$ contains $\mathbf{Z}$.

Problem 3 Give a construction of an Eilenberg-MacLane space $K(\pi, n)$. Prove that

$$H_{n+1}(K(\pi, n); \mathbf{Z}) = 0$$

if $n \geq 2$ and π is an arbitrary abelian group.

Problem 4 Let $f : S^n \times S^n \to S^{2n}$ be the quotient map collapsing $S^n \vee S^n$ to a point. Show that f induces the zero map on all homotopy groups but f is not nullhomotopic.

Problem 5 Let $f : S^2 \to K(\mathbf{Z}, 2)$ be a Serre fiber bundle with a fiber F. Assume that the induced homomorphism $f_* : \pi_2 S^2 \to \pi_2 K(\mathbf{Z}, 2)$ is an isomorphism. Prove that the fiber F is homotopy equivalent to S^3.

Problem 6 Compute the homotopy groups $\pi_q(\mathbf{CP}^n)$ for $q \leq 2n+1$.

Problem 7 Let $f : S^{2n} \to S^{2n}$ be a map of degree zero. Prove that there exists a point $x \in S^{2n}$ with $f(x) = x$ and a point $y \in S^{2n}$ such that $f(y) = -y$.

Problem 8 Let M be a closed, simply-connected manifold of dimension $4k+2$. Show that the Euler characteristic of M is even.

Problem 9 Let $p : E \to B$ be a Serre fiber bundle, where B is a path connected space. Prove that for any two points $x_0, x_1 \in B$ the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent.

Problem 10 State the Lefschetz Fixed Point Theorem. Prove that any map

$$f : \mathbf{CP}^{4k} \times \mathbf{RP}^{2n} \to \mathbf{CP}^{4k} \times \mathbf{RP}^{2n}$$

always has a fixed point. Give an example of a map

$$g : \mathbf{CP}^{4k} \times \mathbf{RP}^{2n-1} \to \mathbf{CP}^{4k} \times \mathbf{RP}^{2n-1}$$

with no fixed points.