# ANALYSIS QUALIFYING EXAM FOR DECEMBER 2020 OR JANUARY 2010

Instructions: Partial credit will be given when appropriate. The decision on this examination will be based not only on the total point score, but also on whether answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot be provided. A completely correct solution is worth more than the same number of points earned in small amounts of partial credit on several problems.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

**Problem 1.** Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $f\colon X \to \mathbb{R}$ be a measurable function on $X$. Show that, for any $\varepsilon > 0$, there exist $M > 0$ and a measurable function $g$ with $|g(x)| \leq M$ for all $x \in X$ such that

$$\mu\big(\big\{x \in X\colon f(x) \neq g(x)\big\}\big) < \varepsilon.$$

Topic: Measure theory.

Difficulty rating: 1 (scale: 1–6).

*Proof.* For $k = 1, 2, \ldots$, set

$$E_k = \big\{x \in X\colon k - 1 \leq |f(x)| < k\big\}.$$

We have $X = \bigcup_{k=1}^{\infty} E_k$ and $E_j \cap E_k = \varnothing$ when $j \neq k$. Therefore $\mu(X) = \sum_{k=1}^{\infty} \mu(E_k)$. Since $\mu(X) < \infty$, there is $M \in \mathbb{Z}_{>0}$ such that $\sum_{k=M+1}^{\infty} \mu(E_k) < \varepsilon$. Define $g(x) = f(x)$ if $x \in \bigcup_{k=1}^{M} E_k$ and $g(x) = 0$ if $x \notin \bigcup_{n=1}^{M} E_k$. Then $|g(x)| \leq M$ for all $x \in X$. Moreover

$$\big\{x \in X\colon f(x) \neq g(x)\big\} = \bigcup_{k=M+1}^{\infty} E_k.$$

Thus

$$\mu\big(\big\{x \in X\colon f(x) \neq g(x)\big\}\big) = \mu\left(\bigcup_{k=M+1}^{\infty} E_k\right) = \sum_{k=M+1}^{\infty} \mu(E_k) < \varepsilon.$$

This completes the solution. □

**Problem 2.** Let $m$ be Lebesgue measure on $\mathbb{R}$. Let $f$ be a function in $L^1([-1, 2], m)$. Show that, for any $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that for any $t \in \mathbb{R}$ with $|t| < \delta$,

$$\int_{[0,1]} \big|f(x + t) - f(x)\big| \, dm(x) < \varepsilon.$$

Topic: Measure theory.

Difficulty rating: 1 (scale: 1–6).

This problem seems too simple.

*Solution.* Define $g\colon \mathbb{R} \to \mathbb{C}$ by

$$g(x) = \begin{cases} f(x) & x \in [-1,2] \\ 0 & x \in \mathbb{R} \setminus [-1,2]. \end{cases}$$

Then $g \in L^1(\mathbb{R})$. By Theorem 9.5 in Rudin's book, the function from $\mathbb{R}$ to $L^1(\mathbb{R})$ which sends $t \in \mathbb{R}$ to the translate of $g$ by $t$, that is, the function $x \mapsto g(x-t)$, is uniformly continuous. In particular, for all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for any $t \in \mathbb{R}$ with $|t| < \delta_0$,

$$\int_{\mathbb{R}} \big| g(x-t) - g(x) \big| \, dm(x) < \varepsilon.$$

Taking $\delta = \min\left(\frac{1}{2}, \delta_0\right)$ gives $\delta \in (0,1)$ and

$$\int_{[0,1]} \big| f(x+t) - f(x) \big| \, dm(x) < \varepsilon.$$

whenever $|t| < \delta$. □

*Alternate solution.* By Theorem 3.14 in Rudin's book, there is a continuous function $g \in C([-1,2])$ such that

$$\|f - g\|_1 = \int_{[-1,2]} |f - g| \, dm < \frac{\varepsilon}{3}.$$

Choose $\delta_0 > 0$ such that, for any $x, y \in [-1,2]$ with $|x - y| < \delta_0$,

$$|g(x) - g(y)| < \frac{\varepsilon}{3}.$$

Set $\delta = \min\left(\frac{1}{2}, \delta_0\right)$. Then

$$\begin{aligned} &\int_{[0,1]} |f(x+\delta) - f(x)| \, dm(x) \\ &\qquad \leq \int_{[0,1]} |f(x+\delta) - g(x+\delta)| \, dm(x) + \int_{[0,1]} |g(x+\delta) - g(x)| \, dm(x) \\ &\qquad\qquad + \int_{[0,1]} |g(x) - f(x)| \, dm(x) \\ &\qquad \leq \|f-g\|_1 + \frac{\varepsilon}{3} + \|f-g\|_1 < \varepsilon. \end{aligned}$$

This completes the solution. □

One may also use Lusin's Theorem directly.

**Problem 3.** Let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of nondecreasing functions which are absolutely continuous on $[a,b]$. Suppose that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in [a,b]$. Show that $f$ is absolutely continuous on $[a,b]$.

Topic: Measure theory.

Difficulty rating: 2 (scale: 1–6).

To get $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$ for almost all $x \in [0,1]$, absolute continuity is not needed. But this is a much harder problem.

*Solution.* For $x \in [a, b]$ and $n \in \mathbb{Z}_{\geq 0}$ define $g_n(x) = f_n(x) - f_n(a)$. Then $(g_n)_{n \in \mathbb{Z}_{>0}}$ is a sequence of nondecreasing functions which are absolutely continuous on $[a, b]$, and for all $x \in [a, b]$ we have

$$\sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{\infty} f_n(a).$$

In particular, the function $g(x) = \sum_{n=1}^{\infty} g_n(x)$ is the sum of a convergent series for all $x \in [a, b]$, and $g$ differs from $f$ by a constant. Therefore it suffices to prove that $g$ is absolutely continuous on $[a, b]$.

We have $g_n(a) = 0$ for all $n \in \mathbb{Z}_{>0}$, and $g_n(x) \geq 0$ for all $n \in \mathbb{Z}_{>0}$ and all $x \in [a, b]$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{Z}_{>0}$ such that

$$\sum_{n=N+1}^{\infty} g_n(b) < \frac{\varepsilon}{2}.$$

Since $g_n$ is absolutely continuous for $n = 1, 2, \ldots, N$, we can choose $\delta_n > 0$ such that whenever $m \in \mathbb{Z}_{>0}$ and

$$(\alpha_1, \beta_1),\, (\alpha_2, \beta_2),\, \ldots,\, (\alpha_m, \beta_m) \subset [a, b]$$

are disjoint open intervals with $\sum_{k=1}^{m} (\beta_k - \alpha_k) < \delta_n$, then

$$\sum_{k=1}^{m} \big[ g_n(\beta_k) - g_n(\alpha_k) \big] < \frac{\varepsilon}{2N}.$$

Set $\delta = \min(\delta_1, \delta_2, \ldots, \delta_m)$. Let

$$(\alpha_1, \beta_1),\, (\alpha_2, \beta_2),\, \ldots,\, (\alpha_m, \beta_m) \subset [a, b]$$

be disjoint open intervals with $\sum_{k=1}^{m} (\beta_k - \alpha_k) < \delta$. Since the functions $g_n$ are nondecreasing and $g_n(a) = 0$, we have

$$\sum_{k=1}^{m} \big[ g_n(\beta_k) - g_n(\alpha_k) \big] \leq g_n(b) - g_n(a) = g_n(b).$$

Therefore

$$\begin{aligned}
\sum_{k=1}^{m} \big[ g(\beta_k) - g(\alpha_k) \big] &= \sum_{k=1}^{m} \left( \sum_{n=1}^{\infty} g_n(\beta_k) - \sum_{n=1}^{\infty} g_n(\alpha_k) \right) = \sum_{k=1}^{m} \sum_{n=1}^{\infty} \big[ g_n(\beta_k) - g_n(\alpha_k) \big] \\
&\leq \sum_{n=1}^{N} \sum_{k=1}^{m} \big[ g_n(\beta_k) - g_n(\alpha_k) \big] + \sum_{n=N+1}^{\infty} g_n(b) \\
&< N \left( \frac{\varepsilon}{2N} \right) + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This completes the solution. □

*Alternate solution.* For $n \in \mathbb{Z}_{>0}$, since $f_n$ is absolutely continuous, for every $x \in [a, b]$ we have

$$f_n(x) = f_n(a) + \int_a^x f_n' \, dm.$$

Since the series

$$\sum_{n=1}^{\infty} f_n(a) \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left( f_n(a) + \int_a^x f_n' \, dm \right) = \sum_{n=1}^{\infty} f_n(x)$$

converge, so does the series $\sum_{n=1}^{\infty} \int_a^x f_n' \, dm$.

Write

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n(a) + \sum_{n=1}^{\infty} \int_a^x f_n' \, dm.$$

For $n \in \mathbb{Z}_{>0}$, we have $f_n' \geq 0$ since $f_n$ is nondecreasing. By the Monotone Convergence Theorem,

$$\sum_{n=1}^{\infty} \int_a^x f_n' \, dm = \int_a^x \left( \sum_{n=1}^{\infty} f_n' \right) dm.$$

Now

$$\sum_{n=1}^{\infty} \int_a^b f_n' \, dm \leq \sum_{n=1}^{\infty} \big( f_n(b) - f_n(a) \big) = \sum_{n=1}^{\infty} f_n(b) - \sum_{n=1}^{\infty} f_n(a) < \infty,$$

so $\sum_{n=1}^{\infty} f_n'$ is integrable on $[a, b]$. Moreover,

$$f(x) = \sum_{n=1}^{\infty} f_n(a) + \int_a^x \sum_{n=1}^{\infty} f_n' \, dm,$$

so $f$ is the indefinite integral of an $L^1$ function and therefore absolutely continuous.

$\square$

**Problem 4.** For $k \in \mathbb{Z}_{>0}$ let $m_k$ denote Lebesgue measure on $\mathbb{R}^k$. Let $r, s \in \mathbb{Z}_{>0}$, and let $f \in L^1(\mathbb{R}^r, m_r)$ and let $g \in L^1(\mathbb{R}^s, m_s)$ be real valued functions. Define $F \colon \mathbb{R}^{rs} \to \mathbb{R}$ by $h(x, y) = f(x)g(y)$ for $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. Prove that $h \in L^1(\mathbb{R}^{rs}, m_{rs})$.

Topic: Measure theory.

Difficulty rating: 1 (scale: 1–6).

The case $r = s = 1$ is implicit in the proof that the convolution of two functions in $L^1(\mathbb{R})$ is again in $L^1(\mathbb{R})$.

The following is a very similar problem. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces, and let $f \in L^1(X, \mathcal{M}, \mu)$ and $g \in L^1(Y, \mathcal{N}, \nu)$. Show that $(x, y) \mapsto f(x)g(y)$ is in $L^1(X \times Y, \, \mu \times \nu)$.

*Solution.* We first prove this assuming that both $f$ and $g$ are nonnegative. Define $F, G \colon \mathbb{R}^{rs} \to \mathbb{R}$ by

$$F(x, y) = f(x) \qquad \text{and} \qquad G(x, y) = g(y)$$

for $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. For $\alpha \in \mathbb{R}$, we have

$$\big\{ z \in \mathbb{R}^{rs} \colon F(z) > \alpha \big\} = \big\{ x \in \mathbb{R}^r \colon f(x) > \alpha \big\} \times \mathbb{R}^s,$$

which is a measurable rectangle because $f$ is measurable. Thus, $F$ is measurable. Similarly, measurability of $g$ implies measurability of $G$. Therefore $h$ is the product of two measurable functions, and hence measurable.

Apply the version of Fubini's Theorem for nonnegative functions, getting

$$\begin{aligned}\int_{\mathbb{R}^r\times\mathbb{R}^s} h\,dm_{rs} &= \int_{\mathbb{R}^r}\left(\int_{\mathbb{R}^s} f(x)g(y)\,dm_s(y)\right)dm_r(x)\\ &= \int_{\mathbb{R}^r} f(x)\left(\int_{\mathbb{R}^s} g(y)\,dm_s(y)\right)dm_r(x)\\ &= \left(\int_{\mathbb{R}^r} f(x)\,dm_r(x)\right)\left(\int_{\mathbb{R}^s} g(y)\,dm_s(y)\right) < \infty.\end{aligned}$$

So $h \in L^1(\mathbb{R}^{rs}, m_{rs})$.

For the general case, let $f = f_+ - f_-$ and $g = g_+ - g_-$ be the usual decomposition of $f$ and $g$ into positive and negative parts. Then $f_+, f_- \in L^1(\mathbb{R}^r, m_r)$ and $g_+, g_- \in L^1(\mathbb{R}^s, m_s)$. It follows from the case already done that the functions assigning to $(x, y)$ the values $f_+(x)g_+(y)$, $f_+(x)g_-(y)$, $f_-(x)g_+(y)$ and $f_-(x)g_+(y)$ are all in $L^1(\mathbb{R}^{rs}, m_{rs})$. Since

$$h(x, y) = f_+(x)g_+(y) - f_-(x)g_+(y) - f_-(x)g_-(y) + f_-(x)g_-(y),$$

it follows that $h$ is in $L^1(\mathbb{R}^{rs}, m_{rs})$. □

A solution which says nothing about measurability of $F$ gets little credit.

*Alternate solution.* Prove that $h$ is measurable as in the first part of the first solution. (The proof given does not use nonnegativity of $f$ and $g$.)

Then use the second part of the proof of the nonnegative case to show that the function $(x, y) \mapsto |f(x)|\cdot|g(y)|$ is integrable. This function is $|h|$, so we are done. □

**Problem 5.** Let $C([0, 1])$ be the real Banach space of real valued continuous functions with the usual supremum norm. For each $f \in C([0, 1])$, define

$$T(f)(x) = \int_0^x f(t)\,dt.$$

Show that $T$ is a bounded linear map from $C([0, 1])$ to itself (except that you need not prove linearity). Then show (this is the main part) that $T$ maps every bounded set to a set whose closure is compact.

Topic: Functional analysis.

Difficulty rating: 2 (scale: 1–6).

Caution is required with this problem. The main point is the use of the Arzela-Ascoli Theorem. Many times Math 616–618 is taught, this theorem is never mentioned. It is supposed to be in Math 413–415, but the 2018–2019 version of Math 413–415 used a different book, which does not contain this theorem. I don't know if the Arzela-Ascoli Theorem was in the course.

*Solution.* Let $f \in C([0, 1])$. Then $T(f)$ is differentiable, hence continuous, so in $C([0, 1])$. We estimate, for all $x \in [0, 1]$,

$$|T(f)(x)| = \int_0^x |f(t)|\,dt \leq \max\bigl(\bigl\{|f(t)| \colon t \in [0, 1]\bigr\}\bigr)\int_0^x dt \leq \|f\|_\infty x \leq \|f\|_\infty.$$

It follows that $\|T\| \leq 1$.

Let $S \subset C([0,1])$ be bounded. Choose $M \in [0,\infty)$ such that $\|f\| \leq M$ for all $f \in S$. Since $\|T\| \leq 1$, $\|T(f)\| \leq M$ for all $f \in S$. Then, for any $f \in S$ and $x, y \in [0,1]$ with $x < y$,

$$|T(f)(x) - T(f)(y)| \leq \int_x^y |f(t)|\,dt \leq M(y-x).$$

It follows that $T(S)$ is equicontinuous. Therefore, by the Arzela-Ascoli Theorem, $T(S)$ has compact closure. □

**Problem 6.** Let $a, b \in \mathbb{R}$ satisfy $a < b$, and let $C([a,b])$ be, as usual, the Banach space of continuous functions from $[a,b]$ to $\mathbb{C}$ with the supremum norm $\|\cdot\|_\infty$. Recall that a sequence $(f_n)_{n \in \mathbb{Z}_{>0}}$ in $C([a,b])$ converges weakly to a function $f \in C([a,b])$ if, for any bounded linear functional $\omega$ on $C([a,b])$, we have $\lim_{n\to\infty} \omega(f_n) = \omega(f)$. Show that a sequence $(f_n)_{n \in \mathbb{Z}_{>0}}$ converges to $f$ weakly if and only if $(\|f_n\|)_{n \in \mathbb{Z}_{>0}}$ is a bounded sequence and, for any $t \in [a,b]$, $\lim_{n\to\infty} f_n(t) = f(t)$.

Topic: Functional analysis.

Difficulty rating: 3 (scale: 1–6).

This uses a lot of machinery, but in a fairly direct way: the Riesz Representation Theorem for bounded linear functionals on $C(X)$, the Radon-Nikodym Theorem, the Dominated Convergence Theorem, the Hahn-Banach Theorem, and the Uniform Boundedness Principle.

*Solution.* Let $E = C([a,b])^*$ be the Banach space dual of $C([a,b])$.

First assume $(\|f_n\|)_{n \in \mathbb{Z}_{>0}}$ is bounded and $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in [a,b]$. Let $\omega \in E$. By the Riesz Representation Theorem, there is a complex measure $\mu$ on $[a,b]$ such that $\omega(g) = \int_{[a,b]} g\,d\mu$ for all $g \in C([a,b])$. Let $h$ be the Radon-Nikodym derivative

$$h = \left[\frac{d\mu}{d|\mu|}\right].$$

Then $|h| = 1$ almost everywhere (this is Theorem 6.12 in Rudin's book) and for all $g \in C([a,b])$ we have

$$\omega(g) = \int_{[a,b]} g\,d\mu = \int_{[a,b]} gh\,d|\mu|.$$

Set $M = \sup_{n \in \mathbb{Z}_{>0}} \|f_n\|$. Then $|f_n h| \leq M$, the constant function $M$ is integrable with respect to $|\mu|$, and $f_n h \to fh$ pointwise almost everywhere with respect to $|\mu|$. Therefore the Dominated Convergence Theorem gives the middle step in the calculation

$$\lim_{n\to\infty} \omega(f_n) = \lim_{n\to\infty} \int_{[a,b]} f_n h\,d|\mu| = \int_{[a,b]} fh\,d|\mu| = \omega(f),$$

as desired.

For the reverse direction, assume $f_n \to f$ weakly. For $t \in [a,b]$, the map $\omega\colon C([a,b]) \to \mathbb{C}$, given by $\omega(g) = g(t)$ for all $g \in C([a,b])$, is clearly a bounded linear functional, with $\|\omega\| = 1$. Therefore $\lim_{n\to\infty} f_n(t) = f(t)$.

It remains to prove that $(\|f_n\|)_{n \in \mathbb{Z}_{>0}}$ is bounded. For any $g \in C([a,b])$, let $g^{**}\colon E \to \mathbb{C}$ be the linear functional given by $g^{**}(\omega) = \omega(g)$ for all $\omega \in E$. Then $\|g^{**}\| = \|g\|$, by Remark 5.21 in Rudin's book. (This is an easy consequence of the Hahn-Banach Theorem.) For all $\omega \in E$, the sequence $(f_n^{**}(\omega))_{n \in \mathbb{Z}_{>0}}$ converges to $\omega(f) = f^{**}(\omega)$, so this sequence is bounded. Now the Banach-Steinhaus Theorem (Uniform Boundedness Principle; Theorem 5.8 of Rudin's book) implies that $\sup_{n \in \mathbb{Z}_{>0}} \|f_n^{**}\| < \infty$, that is, $\sup_{n \in \mathbb{Z}_{>0}} \|f_n\| < \infty$. $\square$

The argument given in the proof shows that the Dominated Convergence Theorem is valid for complex measures and uniformly bounded sequences which converge pointwise almost everywhere. But no such theorem is stated in Rudin's book, so, unless it was officially in the course for the year of the exam, the argument must be given.

**Problem 7.** Let $\Omega \subset \mathbb{C}$ be a connected nonempty open set. Let $f \colon \Omega \to \mathbb{C}$ be a holomorphic function such that $\overline{f}$ is also holomorphic. Prove that $f$ is constant.

Topic: Complex analysis.

Difficulty rating: 3 (scale: 1–6).

Tricky. One solution uses the Cauchy-Riemann equations, which were in this year's course (Spring 2020), but are not always in the course.

An easier version would assume $\Omega = \mathbb{C}$, or $\Omega$ is a rectangle. (This helps with the solution using the Cauchy-Riemann equations, but not with the intended solution.)

*Solution.* The function $g(z) = f(z) + \overline{f}(z)$ is holomorphic on $\Omega$. Its range is contained in $\mathbb{R} \subset \mathbb{C}$, and is therefore not open. Since $\Omega \subset \mathbb{C}$ is connected, the Open Mapping Theorem (the one in complex analysis) implies that $g$ is constant.

The function $h(z) = f(z) - \overline{f}(z)$ is holomorphic on $\Omega$. Its range is contained in $i\mathbb{R} \subset \mathbb{C}$, and is therefore not open. For the same reason as for $g$, this function is also constant.

Therefore $f = \frac{1}{2}(g + h)$ is constant. $\square$

The following alternate solution is valid if the Cauchy-Riemann equations were in the course. Even if they were not, the direction used (a holomorphic function satisfies the Cauchy-Riemann equations) is easy to prove.

*Alternate solution.* Let $U \subset \mathbb{R}^2$ be the set of all pairs $(x, y) \in \mathbb{R}^2$ such that $x + iy \in \Omega$. Set $u(x, y) = \mathrm{Re}(f(x + iy))$ and $v(x, y) = \mathrm{Im}(f(x + iy))$ for $(x, y) \in U$. Then the Cauchy-Riemann equations must hold, that is:

$$D_1 u(x, y) = D_2 v(x, y) \qquad \text{and} \qquad D_2 u(x, y) = -D_1 v(x, y).$$

Applying them to $\overline{f}$ gives:

$$D_1 u(x, y) = -D_2 v(x, y) \qquad \text{and} \qquad D_2 u(x, y) = D_1 v(x, y).$$

It follows that all four partial derivatives

$$D_1 u(x, y), \qquad D_2 u(x, y), \qquad D_1 v(x, y), \qquad \text{and} \qquad D_2 v(x, y),$$

are zero for all $(x, y) \in U$.

For $a, b \in \mathbb{R}$, set

$$V_{a,b} = \big\{ (x, y) \in U \colon u(x, y) = a \text{ and } v(x, y) = b \big\}.$$

We claim that $V_{a,b}$ is open. Let $(x_0, y_0) \in V_{a,b}$, and choose $r > 0$ such that the set

$$R = (x_0 - r,\, x_0 + r) \times (y_0 - r,\, y_0 + r)$$

is contained in $U$. Let $(x, y) \in R$. Applying the Mean Value Theorem to the functions $y \mapsto u(x_0, y)$ and $y \mapsto v(x_0, y)$ using $D_1 u(x_0, y) = 0$ and $D_1 v(x_0, y) = 0$

for $y \in (y_0 - r,\, y_0 + r)$, we see that $u(x_0, y) = u(x_0, y_0) = a$ and $v(x_0, y) = u(x_0, y_0) = b$. A similar argument with the functions $x \mapsto u(x, y)$ and $y \mapsto v(x, y)$ for $x \in (x_0 - r,\, x_0 + r)$ shows that $u(x, y) = u(x_0, y) = a$ and $v(x, y) = u(x_0, y) = b$. Thus $(x, y) \in V_{a,b}$. So $R \subset V_{a,b}$. The claim follows.

Now fix $(x_0, y_0) \in U$. The sets

$$V_{u(x_0,y_0),\, v(x_0,y_0)} \qquad \text{and} \qquad \bigcup_{(a,b) \neq (u(x_0,y_0),\, v(x_0,y_0))} V_{a,b}$$

are disjoint, open, and cover $U$. By connectedness of $U$, one of them, necessarily $V_{u(x_0,y_0),\, v(x_0,y_0)}$, is equal to $U$. Therefore $u$ and $v$ are constant, so $f$ is constant. □

It is simpler to prove that $f$ is constant if $\Omega$ is, say, convex. For general $\Omega$, however, for fixed $x_0 \in \mathbb{R}$, there is no reason for $\{y \in \mathbb{R} \colon x_0 + iy \in \Omega\}$ to be connected. The main point here is that in a connected open set in $\mathbb{R}^2$, any two points can be joined by a piecewice linear path in which all pieces are parallel to one of the coordinate axes, and $u$ and $v$ are constant along any path parallel to one of the coordinate axes.

A standard result in partial differential equations says that if $g$ is a $C^1$ function on an open set in $\mathbb{R}^n$ and $\nabla g = 0$, then $g$ is locally constant. However, there is nothing about partial differential equations in the material this exam is based on.

**Problem 8.** Let $\Omega \subset \mathbb{C}$ be a connected open set, let $a \in \Omega$, let $f \colon \Omega \setminus \{a\} \to \mathbb{C}$ be a holomorphic function which has an essential singularity at $a$, and let $g \colon \Omega \to \mathbb{C}$ be a nonconstant holomorphic function. Prove that the function $h(z) = f(z)g(z)$ has an essential singularity at $a$.

Topic: Complex analysis.

Difficulty rating: 2 (scale: 1–6).

Easier version: Prove that $h$ does not have a removable singularity at $a$.

*Solution.* Suppose first that $h$ has a removable singularity at $a$. Thus there is a holomorphic function $h_0 \colon \Omega \to \mathbb{C}$ such that $h_0(z) = h(z)$ for all $z \in \Omega \setminus \{a\}$. Also, since $g$ is not the constant function 0, here are $N \in \mathbb{Z}_{\geq 0}$ and a holomorphic function $g_0 \colon \Omega \to \mathbb{C}$ such that $g(z) = (z - a)^N g_0(z)$ for all $z \in \Omega$ and such that $g(a) \neq 0$. There is $r > 0$ such that $g_0(z) \neq 0$ for all $z$ in the set $B = \{z \in \mathbb{C} \colon |z - a| < r\}$. Then $z \mapsto h_0(z)/g_0(z)$ is holomorphic on $B$, so there are $c_0, c_1, c_2, \ldots \in \mathbb{C}$ such that

$$\frac{h_0(z)}{g_0(z)} = \sum_{n=0}^{\infty} c_n (z - a)^n$$

for all $z \in B$. Set

$$k(z) = \sum_{n=0}^{\infty} c_{n+N} (z - a)^n$$

for $z \in B$. The series converges on $B$, so $k$ is holomorphic there. The expression, valid for $z \in B \setminus \{a\}$,

$$f(z) = \frac{h(z)}{g(z)} = \frac{h_0(z)}{g_0(z)(z - a)^m} = k(z) + \sum_{n=1}^{N} \frac{c_{N-n}}{(z - a)^n}$$

shows that $f$ has a pole at $a$, which is a contradiction.

Now suppose that $h$ has a pole at $a$. Let $n$ be the order of this pole. Then the function $l(z) = (z-a)^n h(z)$ has a removable singularity at $a$. Apply the previous case with $z \mapsto (z-a)^n g(z)$ in place of $g$. This gives a contradiction.

Since $h$ has neither a removable singularity nor a pole at $a$, it follows that $h$ has an essential singularity at $a$. □

**Problem 9.** Let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on $B_2(0)$. For $n \in \mathbb{Z}_{>0}$ let $Z(n)$ be the number of zeros of $f_n$ in $B_1(0)$, counting with multiplicity. Suppose that there is a function $f \colon B_2(0) \to \mathbb{C}$ such that $f_n \to f$ uniformly on compact subsets of $B_2(0)$, that none of the functions $f_n$ is the zero function, and that $f(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = 1$. Prove that $\sup_{n \in \mathbb{Z}_{>0}} Z(n)$ is finite.

Topic: Complex analysis.

Difficulty rating: 2 (scale: 1–6).

*Solution.* We will prove that in fact $\lim_{n \to \infty} Z(n)$ exists. Since $Z(n)$ is finite for all $n \in \mathbb{Z}_{>0}$, this implies the result.

Since $f_n$ is holomorphic for all $n$ and $f_n \to f$ uniformly on compact subsets of $B_2(0)$, it follows that $f$ is holomorphic.

Since the set $S = \{z \in \mathbb{C} \colon |z| = 1\}$ is compact and $f$ does not vanish on this set, there is $r > 0$ such that $|f(z)| \geq r$ for all $z \in S$. Choose $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ we have

$$\sup_{z \in S} |f_n(z) - f(z)| < \frac{r}{2}.$$

Then for all $n \geq N$ and $z \in S$ we have $|f_n(z) - f(z)| < |f(z)|$. It follows from Rouche's Theorem that $f_n$ and $f$ have the same number of zeros in $B_1(0)$, counting with multiplicity. In particular, all values of $Z(n)$ for $n \geq N$ are equal. □

*Alternate solution.* This solution does not use the assumption that $f$ is holomorphic.

We will prove that in fact $\lim_{n \to \infty} Z(n)$ exists. Since $Z(n)$ is finite for all $n \in \mathbb{Z}_{>0}$, this implies the result.

Since the set $S = \{z \in \mathbb{C} \colon |z| = 1\}$ is compact and $f$ does not vanish on this set, there is $r > 0$ such that $|f(z)| \geq r$ for all $z \in S$. Choose $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ we have

$$\sup_{z \in S} |f_n(z) - f(z)| < \frac{r}{4}.$$

Then for all $n \geq N$ and $z \in S$ we have

$$|f_n(z)| > |f(z)| - \frac{r}{4} \geq \frac{3r}{4}.$$

So for all $m, n \geq N$ and $z \in S$ we have

$$|f_m(z) - f_n(z)| \leq |f_m(z) - f_n(z)| + |f_m(z) - f_n(z)| < \frac{r}{4} + \frac{r}{4} < \frac{3r}{4} < |f_n(z)|.$$

It follows from Rouche's Theorem that $f_m$ and $f_n$ have the same number of zeros in $B_1(0)$, counting with multiplicity. In particular, all values of $Z(n)$ for $n \geq N$ are equal. □