ANALYSIS QUALIFYING EXAM FOR DECEMBER 2020 OR JANUARY 2010

Instructions: Partial credit will be given when appropriate. The decision on this examination will be based not only on the total point score, but also on whether answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot be provided. A completely correct solution is worth more than the same number of points earned in small amounts of partial credit on several problems.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

Problem 1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $f \colon X \to \mathbb{R}$ be a measurable function on X. Show that, for any $\varepsilon > 0$, there exist $M > 0$ and a measurable function g with $|g(x)| \leq M$ for all $x \in X$ such that

$$\mu\big(\big\{x \in X \colon f(x) \neq g(x)\big\}\big) < \varepsilon.$$

Problem 2. Let m be Lebesgue measure on $\mathbb{R}$. Let f be a function in $L^1([-1,2], m)$. Show that, for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that for any $t \in \mathbb{R}$ with $|t| < \delta$,

$$\int_{[0,1]} \big|f(x+t) - f(x)\big| \, dm(x) < \varepsilon.$$

Problem 3. Let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of nondecreasing functions which are absolutely continuous on $[a,b]$. Suppose that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in [a,b]$. Show that f is absolutely continuous on $[a,b]$.

Problem 4. For $k \in \mathbb{Z}_{>0}$ let m_k denote Lebesgue measure on $\mathbb{R}^k$. Let $r, s \in \mathbb{Z}_{>0}$, and let $f \in L^1(\mathbb{R}^r, m_r)$ and let $g \in L^1(\mathbb{R}^s, m_s)$ be real valued functions. Define $F \colon \mathbb{R}^{rs} \to \mathbb{R}$ by $h(x,y) = f(x)g(y)$ for $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. Prove that $h \in L^1(\mathbb{R}^{rs}, m_{rs})$.

Problem 5. Let $C([0,1])$ be the real Banach space of real valued continuous functions with the usual supremum norm. For each $f \in C([0,1])$, define

$$T(f)(x) = \int_0^x f(t)\, dt.$$

Show that T is a bounded linear map from $C([0,1])$ to itself (except that you need not prove linearity). Then show (this is the main part) that T maps every bounded set to a set whose closure is compact.

Problem 6. Let $a, b \in \mathbb{R}$ satisfy $a < b$, and let $C([a,b])$ be, as usual, the Banach space of continuous functions from $[a,b]$ to $\mathbb{C}$ with the supremum norm $\|\cdot\|_\infty$. Recall that a sequence $(f_n)_{n \in \mathbb{Z}_{>0}}$ in $C([a,b])$ converges weakly to a function $f \in C([a,b])$ if, for any bounded linear functional ω on $C([a,b])$, we have $\lim_{n \to \infty} \omega(f_n) = \omega(f)$. Show that a sequence $(f_n)_{n \in \mathbb{Z}_{>0}}$ converges to f weakly if and only if $(\|f_n\|)_{n \in \mathbb{Z}_{>0}}$ is a bounded sequence and, for any $t \in [a,b]$, $\lim_{n \to \infty} f_n(t) = f(t)$.

Problem 7. Let $\Omega \subset \mathbb{C}$ be a connected nonempty open set. Let $f \colon \Omega \to \mathbb{C}$ be a holomorphic function such that $\overline{f}$ is also holomorphic. Prove that f is constant.

Problem 8. Let $\Omega \subset \mathbb{C}$ be a connected open set, let $a \in \Omega$, let $f \colon \Omega \setminus \{a\} \to \mathbb{C}$ be a holomorphic function which has an essential singularity at a, and let $g \colon \Omega \to \mathbb{C}$ be a nonconstant holomorphic function. Prove that the function $h(z) = f(z)g(z)$ has an essential singularity at a.

Problem 9. Let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on $B_2(0)$. For $n \in \mathbb{Z}_{>0}$ let $Z(n)$ be the number of zeros of f_n in $B_1(0)$, counting with multiplicity. Suppose that there is a function $f \colon B_2(0) \to \mathbb{C}$ such that $f_n \to f$ uniformly on compact subsets of $B_2(0)$, that none of the functions f_n is the zero function, and that $f(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = 1$. Prove that $\sup_{n \in \mathbb{Z}_{>0}} Z(n)$ is finite.