Algebra Qualifying Exam Winter 2021 Solutions

1. Let $\Bbbk$ be a field, and let $a, b \in \Bbbk^\times$ be invertible scalars. Let $A_{a,b}$ be the non-commutative $\Bbbk$-algebra generated by symbols x, y, z, modulo the relations

$$zy = yz + az, \tag{1a}$$

$$yx = xy + bx, \tag{1b}$$

$$zx = xz + y. \tag{1c}$$

Note: Part (a) is difficult to do without knowing certain techniques. An alternative approach for significant partial credit is found at the end of the problem.

a) Prove that

$$A_{a,b} \text{ has a } \Bbbk\text{-basis of the form } \mathbb{B} = \{x^k y^\ell z^m\}_{k,\ell,m \geq 0}, \tag{2}$$

if and only if $a = b$. If $a \neq b$, prove that $A_{a,b} \cong 0$.

Solution: This is an application of the Bergman diamond lemma. Order the generators $x < y < z$. Let us place the deglex order on monomials, which satisfies the DCC. With respect to this order, the relations can be rephrased as reduction rules

$$zy \mapsto yz + az,$$

$$yx \mapsto xy + bx,$$

$$zx \mapsto xz + y.$$

The input words are $\{zy, yx, zx\}$, and a monomial is irreducible (doesn't have an input word as a subword) if and only if it has the form $x^k y^\ell z^m$. Irreducible monomials form a basis if and only if the ambiguities are resolvable, by the Bergman diamond lemma. The only ambiguity in the input words is the overlap ambiguity in the word zyx. Resolving this word one way yields

$$zyx \mapsto zxy + b \cdot zx \mapsto xzy + y^2 + b \cdot xz + b \cdot y \mapsto xyz + a \cdot xz + y^2 + b \cdot xz + b \cdot y.$$

Resolving the word the other way yields

$$zyx \mapsto yzx + a \cdot zx \mapsto yxz + y^2 + a \cdot xz + a \cdot y \mapsto xyz + b \cdot xz + y^2 + a \cdot xz + a \cdot y.$$

The difference between these resolutions is $(a - b) \cdot y$, so the ambiguity is resolvable if and only if $a = b$.

The computation done above implies that $0 = zyx - zyx = (b - a) \cdot y$. If $a \neq b$ then $y = 0$. If $y = 0$ then $az = zy - yz = 0$, so that $z = 0$ (we assume $a \in \Bbbk^\times$). Similarly $x = 0$. Thus $A_{a,b} = 0$.

b) How many isomorphism classes of $\Bbbk$-algebras $A_{a,b}$ are there (still assuming that $a, b \in \Bbbk^\times$)?

Solution: Clearly the 0 algebra is an isomorphism class, accounting for all $A_{a,b}$ with $a \neq b$. When $a = b$, rescaling y and x both by a will give the relations

$$azy = ayz + az$$

$$a^2 yx = a^2 xy + a^2 x$$

$$azx = axz + ay.$$

Dividing by the appropriate power of a, this gives the relations as though $a = b = 1$. So $A_{a,a} \cong A_{1,1}$ by the map which rescales y and x. Hence there are two isomorphism classes (not one, since $A_{a,a} \neq 0$).

c) Let $A = A_{2,2}$, and assume $\Bbbk$ has characteristic zero. There is a homomorphism $\rho\colon A \to \mathrm{End}_{\Bbbk}(\Bbbk[t])$ for which

$$\rho(y)(t^k) = (2k+1)t^k. \tag{3}$$

Let M be an A-submodule of $\Bbbk[t]$, and $f = \sum_i c_i t^i$ be an element of M. Prove that $t^i \in M$ for each $i \geq 0$ such that $c_i \neq 0$.

Solution: By (3), the operator y has distinct eigenvalues for different powers of t. If the highest degree of f is N, then the operator $p = (y-1)(y-3)\cdots\widehat{(y-(2i+1))}\cdots(y-(2N+1))$ will kill any monomial t^k except for $k = i$, and will rescale t^i by an invertible scalar. Thus $p \cdot f \in M$, and is an invertible scalar multiple of t^i.

Partial credit alternative to (a):

Here is the full definition of the action of A on $\Bbbk[t]$.

$$x(t^k) = -kt^{k-1}, \quad x(t^0) = 0. \tag{4a}$$

$$y(t^k) = (2k+1)t^k, \tag{4b}$$

$$z(t^k) = (k+1)t^{k+1}. \tag{4c}$$

Prove that $\mathbb{B}' = \{z^k y^\ell x^m\}_{k,\ell,m\geq 0}$ is a spanning set, and prove that $\{z^k x^m\} \subset \mathbb{B}'$ is linearly independent when $a = b$.

Solution: We prove that $\mathbb{B}'$ spans, by proving that any word in the generators is in its span. We do this by induction on the length of the word. After that we induce on the lexicographic order induced by $z < y < x$. Using the relations, any word of length d which is not in X will be non-minimal in the lexicographic order, and will have a x before an z, or a y before a z, or a x before an y. Using the appropriate relation, we can rewrite this word as a shorter word, plus a word which is smaller in the lexicographic order. By induction, both these terms are in the span of $\mathbb{B}'$.

Now suppose there is a nontrivial linear combination $\alpha = \sum c_{k,m} z^k x^m$ of elements of X which is zero. Up to isomorphism we can assume $a = b = 2$. Thus A acts on $\Bbbk[t]$, and α acts by zero. Let m be minimal such that α has some nonzero coefficient $c_{k,m}$, and let k be maximal among these terms. Only terms with x exponent m can act nontrivially on t^m, and only the terms with k maximal will send t^m to a polynomial with nontrivial coefficient of t^k. Hence, the coefficient of t^k in $\alpha(t^m)$ is $c_{k,m} \neq 0$, a contradiction.

2. Let A be a finite-dimensional algebra over $\mathbb{C}$, and let $Z(A)$ denote the center of A.

a) Suppose L is a finitely-generated irreducible A-module. Prove that any $z \in Z(A)$ acts on L by $\lambda \cdot \mathrm{id}_L$ for some scalar $\lambda \in \mathbb{C}$. (We refer to λ as the eigenvalue of z on L.)

Solution: If L is finitely-generated then L is finite-dimensional over $\mathbb{C}$, as is $\mathrm{End}(L)$. By Schur's Lemma, $\mathrm{End}_A(L) = \mathbb{C}$, since this is the only finite-dimensional division algebra over $\mathbb{C}$. Now the action of $z \in Z(A)$ commutes with the action of any $a \in A$, so z acts by a element of $\mathrm{End}_A(L)$, i.e. a scalar multiple of the identity.

b) Pick $z \in Z(A)$. Let M and N be A-modules for which z acts on M by $\lambda \cdot \mathrm{id}_M$ and z acts on N by $\mu \cdot \mathrm{id}_N$, for $\lambda, \mu \in \mathbb{C}$. Prove that when $\lambda \neq \mu$, any short exact sequence of A-modules of the form

$$0 \to M \xrightarrow{f} X \xrightarrow{g} N \to 0 \tag{5}$$

actually splits.

Solution: We use Lagrange interpolation to produce a (co)splitting, a map $p\colon X \to M$ such that $p \circ f = \mathrm{id}_M$. Any polynomial in z will produce an element of $\mathrm{End}_A(X)$. We claim that $p = (z-\mu)(\lambda-\mu)$ acts as the identity on any element of M, which is clear since z acts by λ on M. We also claim that p has image contained inside M. If $\bar{x}$ represents the image in N of an element $x \in X$, then $\overline{p(x)} = p(\overline{x}) = 0$, so $p(x) \in M$.

c) Continuing the above, find an explicit example where $\lambda = \mu$ and (5) does not split.

Solution: We can let $A = \mathbb{C}[x]/(x^k)$ for $k \geq 2$, $z = x$, $\lambda = \mu = 0$, and the short exact sequence

$$0 \to \mathbb{C} \to X \to \mathbb{C} \to 0$$

where x acts on $X \cong \mathbb{C}^2$ by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

d) When A is semisimple, prove that for any two non-isomorphic irreducible representations L and L', there is an element $z \in Z(A)$ which has distinct eigenvalues on L and on L'.

Solution: By the Artin-Wedderburn theorem, A is isomorphic to a product of matrix algebras over $\mathbb{C}$ (this being the only finite-dimensional division algebra over $\mathbb{C}$). So

$$A \cong \Pi_{i=1}^{r} \mathrm{Mat}_{n_i}(\mathbb{C}),$$

and

$$Z(A) \cong \Pi_{i=1}^{r} Z(\mathrm{Mat}_{n_i}(\mathbb{C})) = \Pi_{i=1}^{r} \mathbb{C} \cdot I.$$

The irreducible modules over A are the vector modules over each matrix algebra. So the identity of the i-th matrix algebra will act by 1 on the i-th irreducible representation, and by zero on the other irreducible representations.

e) Find an example of a finite-dimensional algebra A over $\mathbb{C}$, and two non-isomorphic irreducible modules L and L', such that for any $z \in Z(A)$ the eigenvalue of z on L and on L' agree. (In other words, the center can not distinguish between these irreducible modules.)

Solution: Let A be the algebra of upper triangular 2×2 matrices. This has a one-dimensional module where $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ acts by a, and another one-dimensional module where this matrix acts by d. However, the center of A consists of scalar matrices for which $a = d$.

3. Let A be a finite-dimensional algebra over $\mathbb{C}$. For $a \in A$, let L_a denote left multiplication by a, a linear map $A \to A$. Define a bilinear pairing $A \times A \to \mathbb{C}$ which sends $(a, b) \mapsto \mathrm{trace}(L_a \circ L_b)$. Define the *radical* of A to be

$$\mathrm{rad}\, A := \{a \in A \mid (a, b) = 0 \text{ for all } b \in A\}. \tag{6}$$

a) Prove that $(-, -)$ is symmetric and associative. (A form on an algebra is *associative* if $(ab, c) = (a, bc)$.)

Solution: Note that $L_a \circ L_b = L_{ab}$. Thus $(ab, c) = \mathrm{trace}(L_{abc}) = (a, bc)$. Note also that for any linear operators X and Y one has $\mathrm{trace}(XY) = \mathrm{trace}(YX)$. Thus $(a, b) = \mathrm{trace}(L_a L_b) = \mathrm{trace}(L_b L_a) = (b, a)$.

b) Prove that $\mathrm{rad}\, A$ is a two-sided ideal.

Solution: Suppose that $a \in \mathrm{rad}\, A$ and let $c \in A$ be arbitrary. Then for any $b \in A$ we have $(ac, b) = (a, cb) = 0$, so $ac \in \mathrm{rad}\, A$. Similarly, for any $b \in A$ we have $(ca, b) = (b, ca) = (bc, a) = (a, bc) = 0$, so $ca \in \mathrm{rad}\, A$.

c) Let I be an ideal for which every element is nilpotent. Prove that $I \subset \mathrm{rad}\, A$. Also prove that I is contained in the Jacobson radical of A.

Solution: If a is nilpotent then L_a is nilpotent and has zero trace. If $a \in I$ then for all $b \in A$, ab is also in I so ab is nilpotent and $\mathrm{trace}(L_{ab}) = 0$. Hence $a \in \mathrm{rad}\, A$. Similarly, for all $b \in A$, ab is nilpotent, so $1 + ab$ is invertible (with inverse $1 - ab + (ab)^2 - (ab)^3 + \ldots$). Thus a is in the Jacobson radical.

d) Prove that every element of $\mathrm{rad}\, A$ is nilpotent.

Solution: Let us choose a basis for A such that L_a is expressed by an upper triangular matrix with diagonal entries $(\lambda_1, \ldots, \lambda_n)$. Then $\mathrm{trace}(L_a) = \sum \lambda_i$, and $\mathrm{trace}(L_a^k) = \sum \lambda_i^k$ for all $k \geq 1$. If $a \in \mathrm{rad}\, A$ then $\mathrm{trace}(L_{a^k}) = \mathrm{trace}(L_a^k) = 0$ for all $k \geq 1$. So all power sums in the λ_i vanish, which means all symmetric polynomials in the λ_i vanish, implying that $\lambda_i = 0$ for all i. Hence L_a is nilpotent. (There are various other ways of proving this fact, using Fitting's lemma etc etc.)

Note: There is a theorem which states that an a finite-dimensional algebra, the Jacobson radical is a nilpotent ideal. Hence $\mathrm{rad}\, A = J(A)$.

4. This exercise focuses on the complex representation theory of the finite group $G = GL_2(\mathbb{F}_3)$. Here is its character table: for each conjugacy class we give a representative matrix, and indicate the order of that matrix. Here, I represents the identity matrix.

C	I	$-I$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
$\lvert C\rvert$	1	1	α	8	8	6	β	γ
order	1	2	2	3	6	4	8	8
$\mathbb{1}$	1	1	1	1	1	1	1	1
det	1	1	-1	1	1	1	-1	-1
X	3	3	1	0	0	-1	-1	-1
Y	3	3	-1					
Z_1								
Z_2								
Z_3								
W								

You may freely use the fact that

$(*):\quad$ if A conjugates $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ to an upper triangular matrix, then A itself is upper triangular.

a) Compute the size of the centralizer of $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. What is the size α of the conjugacy class of σ?

Solution: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ centralizes σ then $A\sigma = \sigma A$. Computing this explicitly we get

$$\begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}.$$

Thus A centralizes σ if and only if $b = c = 0$. There are four such matrices, since the diagonal subgroup is isomorphic to $\mathbb{F}_3^\times \times \mathbb{F}_3^\times$.

The size of a conjugacy class is the size of G divided by the size of the centralizer. The size of $GL_2(\mathbb{F}_q)$ in general is $(q^2-1)(q^2-q)$, so when $q = 3$ we get 48. Thus $\alpha = \frac{48}{4} = 12$.

b) Fill in the column of the character table associated to σ.

Solution: By column orthogonality, the dot product of the column with itself is the size of the centralizer, which is 4. Since we already see four nonzero entries ± 1, the dot product can be 4 if and only if the rest of the column is zero.

c) Let $B \subset G$ be the subgroup of upper triangular matrices. Which conjugacy classes have non-empty intersection with B?

Solution: The subgroup B has size $q(q-1)^2$ or 12, since it is a semidirect product $(\mathbb{F}_q^\times)^2 \ltimes \mathbb{F}_q$. Thus it has no elements of order 8. Also, its Sylow 2-subgroup is the diagonal matrices, which has no elements of order 4. These considerations rule out the last three conjugacy classes, and the rest already have representatives in B.

d) There is a one-dimensional representation L of B on which $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ acts by the scalar a. Let $W = \operatorname{Ind}_B^G L$. Use the formula for induced characters to compute the character of W.

Solution: Note that G/B has size 4, so we can choose four representatives $\{A_1, A_2, A_3, A_4\}$, one for each coset. Exactly one of these representatives is in B; we can let it be $A_1 = 1$. Then the formula for induced characters says that

$$\chi_{\mathrm{Ind}\, L}(g) = \sum_i \dot{\chi}_L(A_i g A_i^{-1}),$$

where $\dot{\chi}(h)$ is equal to $\chi(h)$ when $h \in B$ and 0 when $h \notin B$. So we clearly get 0 for the last three conjugacy classes, since they do not intersect B.

For $g = \pm I$, g is central so $A_i g A_i^{-1} = g$ for all i. Thus $\chi_{\mathrm{Ind}\, L}(\pm I) = 4\chi_L(\pm I) = \pm 4$.

For the fourth and fifth conjugacy classes we may use (*) to deduce that only A_1 conjugates the given representative g into B. Thus $\chi_{\mathrm{Ind}\, L}(g) = \chi_L(g)$ is the upper left entry of the matrix.

Finally, conjugating σ by the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ produces $-\sigma$, and $\chi_L(\sigma) + \chi_L(-\sigma) = 0$. All elements in an orbit of a group action are hit an equal number of times, so regardless of whether σ has two or four conjugates inside B, σ and $-\sigma$ will appear an equal number of times, and $\chi_{\mathrm{Ind}\, L}(\sigma) = 0$.

Thus the character of $\mathrm{Ind}_B^G L$ is $(4, -4, 0, 1, -1, 0, 0, 0)$.

e) Explain why W is irreducible.

Solution: Pairing W with itself yields $\frac{1}{48}(4^2 + (-4)^2 + 8(1)^2 + 8(-1)^2 = 1$. Thus W is irreducible.

f) Does L appear as a direct summand of the restricted representation $\mathrm{Res}_G^B X$ or $\mathrm{Res}_G^B Z_i$ for any i? Justify your answer.

Solution: No. By Frobenius reciprocity, the multiplicity of L as a summand of $\mathrm{Res}\, V$ (for a simple module V) is equal to the multiplicity of V as a summand of $\mathrm{Ind}\, L$. Since only W has nontrivial multiplicity inside $\mathrm{Ind}\, L$, no other simple V will have L as a summand of $\mathrm{Res}\, V$.

Note: You do not need to fill in the entire character table.

5. Let B be a $\mathbb{C}$-algebra and $A \subset B$ be a subalgebra.

a) Restriction is a functor R from B-modules to A-modules. Describe R as tensor product with a bimodule. Describe R as Hom with a bimodule.

Solution: $R \cong {}_A B_B \otimes_B (-)$. Note that $B \otimes_B (-)$ is the identity on the underlying vector space. Also, $R \cong \mathrm{Hom}_B({}_B B_A, -)$. Note that $\mathrm{Hom}_B(B, -)$ is the identity on the underlying vector space.

b) Induction is a functor I from A-modules to B-modules. We know that induction and restriction are adjoint functors (which is the left adjoint?). Construct explicitly the unit and counit of adjunction using bimodule morphisms.

Solution: Note that $I := {}_B B \otimes_A (-)$. So $I \circ R$ is tensoring with the (B, B)-bimodule $B \otimes_A B$, and $R \circ I$ is tensoring with the (A, A)-bimodule ${}_A B_A$.

I is left adjoint to R. The unit of adjunction $\mathbb{1}_A \to R \circ I$ is the (A, A)-bimodule map $A \to {}_A B_A$ given by inclusion, $1 \mapsto 1$. The counit of adjunction $I \circ R \to \mathbb{1}_B$ is the (B, B)-bimodule map $B \otimes_A B \to B$ given by multiplication, $1 \otimes 1 \mapsto 1$.

c) Now let $A = \mathbb{C}[x^3] \subset B = \mathbb{C}[x]$. Prove the induction and restriction are adjoint in the other direction as well (i.e. they are biadjoint). The unit of adjunction is the (B, B)-bimodule morphism given on the identity by

$$\Delta \colon B \to B \otimes_A B, \quad 1 \mapsto x^2 \otimes 1 + x \otimes x + 1 \otimes x^2.$$

Verify that Δ extends to a well-defined B-bimodule morphism. Find the counit of adjunction and check the adjunction relations.

Solution: For Δ to be a left B-module map, it must send $f \in \mathbb{C}[x]$ to $fx^2 \otimes 1 + fx \otimes x + f \otimes x^2$. For it to be a right B-module map, it must send f to $x^2 \otimes f + x \otimes fx + 1 \otimes fx^2$. It is enough to check the equality of these two sides for $f = x$, since $x\Delta(1) = \Delta(1)x$ will imply the same for any x^k and then any $f \in \mathbb{C}[x]$. In this case, $x\Delta(1) - \Delta(1)x = x^3 \otimes 1 - 1 \otimes x^3 = 0$, since $x^3 \in A$.

d) The counit of adjunction is an (A, A)-bimodule morphism $\partial \colon B \to A$. Since B is free over A with basis $\{1, x, x^2\}$, ∂ is determined by what it does to this basis.

We need to check that $I \to (IR)I = I(RI) \to I$ is equal to the identity of I, which amounts to checking that

$$(1 \otimes \partial) \circ \Delta \colon B \to B \otimes_A B \to B$$

is the identity map. Applied to $f \in B$ this is the equality

$$fx^2\partial(1) + fx\partial(x) + f\partial(x^2) = f.$$

This is clear from the definition of ∂. (And you could probably have used it to deduce what ∂ was!)

Note: if you wrote $\Delta(f)$ a different way, you might have ended up with the equality

$$\partial(fx^2) + x\partial(fx) + x^2\partial(f) = f.$$

This too you can check directly, perhaps most easily by checking for $f = x^k$ for each k. These two versions of the equation are equivalent, once you've shown that Δ is a B-bimodule map!

Similarly, we need to check that $R \to R(IR) = (RI)R \to R$ is the identity of R, which is just that $(\partial \otimes 1) \circ \Delta = \mathrm{id}_B$. The computation is identical.

e) Prove that the algebra of natural transformations from $R \circ I$ to itself is isomorphic to the algebra $\mathrm{Mat}_3(A)$ of 3×3 matrices with coefficients in A.

Solution: It is clear that B is free as an A-module with basis $\{1, x, x^2\}$. So the endomorphisms of B as an (A, A)-bimodule are the same as the endomorphisms of $A^{\oplus 3}$, which is $\mathrm{Mat}_3(A)$.

6. In this problem, we work over $\mathbb{C}$, and X is the algebraic set inside $\mathbb{A}^2$ cut out by the equation

$$x^2 - 4(y+1)^2 = 1.$$

a) There is a regular function $\varphi\colon X \to \mathbb{A}^1$ which comes from projecting X to the x-axis. Explicitly write down the corresponding homomorphism φ^* of $\mathbb{C}$-algebras, and justify your answer. Please use the t variable for $\mathbb{A}^1$, i.e. $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[t]$.

   Solution: We have $\mathbb{C}[X] = \mathbb{C}[x,y]/(x^2 - 4(y+1)^2 - 1)$. Then $\varphi^*\colon \mathbb{C}[t] \to \mathbb{C}[X]$, and $\varphi^*(t) = x$. This is because composing φ with the coordinate map t will send $(a,b) \mapsto (a) \mapsto a$, and this is exactly what the regular function a does.

b) Give the definition of an integral extension of commutative rings. Is φ^* an integral extension? Prove your answer.

   Solution: If R is a subring of a commutative ring S, then S is integral over R if each element of S satisfies a monic polynomial equation with coefficients in R.

   Yes. It is easy to see that the map φ^* is injective (say, using the Bergman diamond lemma to prove that $\{x^k y^\ell\}_{0\le k, 0\le \ell\le 1}$ is a basis). It has image $\mathbb{C}[x] \subset \mathbb{C}[X]$. We need only show that generators of $\mathbb{C}[X]$ over $\mathbb{C}[x]$ all satisfy a monic polynomial equation with coefficients in $\mathbb{C}[x]$. But $\mathbb{C}[X]$ is generated over the subalgebra $\mathbb{C}[x]$ by y, and $y^2 + 8y + (5 - x^2) = 0$.

c) There is a family of homomorphisms $f_a^*\colon \mathbb{C}[t] \to \mathbb{C}[x,y]$ for $a \in \mathbb{C}$, which sends $t \mapsto x + ay$. Explicitly describe the corresponding regular function $f_a\colon \mathbb{A}^2 \to \mathbb{A}^1$.

   Solution: f_a sends $(b,c) \mapsto (b + ac)$.

d) For which a is f_a^* an integral extension? Prove your answer is correct.

   Solution: Note that the equation defining X is $(x-2y-2)(x+2y+2) = 1$. If $a = \pm 2$ then we claim that f_a^* is not an integral extension. One way to prove this is using the failure of the going up theorem, which amounts to the failure of surjectivity on algebraic sets. If $a = -2$ then it is impossible for $(b,c) \in X$ and $b - 2c = 2$, since then $(b - 2c - 2) = 0$ and $(b-2c-2)(b+2c+2) = 0 \neq 1$.

   If $a \neq \pm 2$ we claim that f_a^* is an integral extension. We have already checked the case when $a = 0$.

   If $a \neq 0, \pm 2$ then $z = x + ay$ and x generate $\mathbb{C}[X]$, and we can rewrite the relation $x^2 - 4(y+1)^2 = 1$ as $x^2 - \frac{4}{a^2}(z - x + a)^2 - 1 = 0$ or as

$$x^2\left(1 - \frac{4}{a^2}\right) + \frac{8}{a^2}x(z+a) - \left(\frac{4}{a^2}(z+a)^2 - 1\right) = 0$$

   which is a polynomial in x with invertible leading coefficient. Thus x is integral over the image of $\mathbb{C}[t]$. Since $\{z^k x^\ell\}_{0\le k, 0\le \ell\le 1}$ is a basis for $\mathbb{C}[X]$, the map f_a^* is injective.