Algebra Qualifying Exam Winter 2021

This is just an intro page so you know what is coming. You don't need to start your clocks until you turn the page.

Take this test in an uninterrupted 3 hour chunk, *without* using any outside resources.

Here is the format of what will come. There are six long problems. I'm not expecting people to finish six problems, but I'm expecting people to finish three problems, and get a smattering of other points.

Each problem will be worth 20 points. In addition, each will be graded with a completion score from 0 to 5. This completion score will then be translated into a numerical score according to the table below. Thus the problems are really worth up to 35 points each. When assigning a completion score, we will typically choose 6 main ideas or tasks which need to be overcome in a problem, and give a point for each one. So, if you miss one, you can still get a score of 5.

Finally, the overall test will be graded with a correctness score from 0 to 5, trying to measure what percentage of the stuff you write is actually correct. This correctness score will then be translated into a numerical score according to the same table.

Good luck!

Test specific notes:

Unless otherwise stated, all representations are finite-dimensional with base field $\mathbb{C}$.

There is a character table in problem 6, parts of which I would like you to fill in. If you are not printing out the test, you are welcome to give yourself extra time to copy down this character table.

For an ideal I and a positive integer $n > 0$, I^n is the ideal $I \cdot I \cdot \ldots \cdot I$.

Converting completion/correctness scores to numerical scores:

- A 0 is worth 0 points.
- A 1 is worth 3 points.
- A 2 is worth 5 points.
- A 3 is worth 7 points.
- A 4 is worth 11 points.
- A 5 is worth 15 points.

Overview of problems:

- Problem 1: algebras defined by generators and relations.
- Problem 2: Artin-Wedderburn, diagonalization.
- Problem 3: Jacobson radical, linear algebra.
- Problem 4: Representation theory, character tables, induction.
- Problem 5: Induction and restriction, adjunction, bimodules.
- Problem 6: Algebraic geometry, integral extensions.

1. Let $\Bbbk$ be a field, and let $a, b \in \Bbbk^\times$ be invertible scalars. Let $A_{a,b}$ be the non-commutative $\Bbbk$-algebra generated by symbols x, y, z, modulo the relations

$$zy = yz + az, \tag{1a}$$

$$yx = xy + bx, \tag{1b}$$

$$zx = xz + y. \tag{1c}$$

Note: Part (a) is difficult to do without knowing certain techniques. An alternative approach for significant partial credit is found at the end of the problem.

a) Prove that

$$A_{a,b} \text{ has a } \Bbbk\text{-basis of the form } \mathbb{B} = \{x^k y^\ell z^m\}_{k,\ell,m\geq 0}, \tag{2}$$

if and only if $a = b$. If $a \neq b$, prove that $A_{a,b} \cong 0$.

b) How many isomorphism classes of $\Bbbk$-algebras $A_{a,b}$ are there (still assuming that $a, b \in \Bbbk^\times$)?

c) Let $A = A_{2,2}$, and assume $\Bbbk$ has characteristic zero. There is a homomorphism $\rho\colon A \to \mathrm{End}_\Bbbk(\Bbbk[t])$ for which

$$\rho(y)(t^k) = (2k+1)t^k. \tag{3}$$

Let M be an A-submodule of $\Bbbk[t]$, and $f = \sum_i c_i t^i$ be an element of M. Prove that $t^i \in M$ for each $i \geq 0$ such that $c_i \neq 0$.

Partial credit alternative to (a):
Here is the full definition of the action of A on $\Bbbk[t]$.

$$x(t^k) = -kt^{k-1}, \quad x(t^0) = 0. \tag{4a}$$

$$y(t^k) = (2k+1)t^k, \tag{4b}$$

$$z(t^k) = (k+1)t^{k+1}. \tag{4c}$$

Prove that $\mathbb{B}' = \{z^k y^\ell x^m\}_{k,\ell,m\geq 0}$ is a spanning set, and prove that $\{z^k x^m\} \subset \mathbb{B}'$ is linearly independent when $a = b$.

2. Let A be a finite-dimensional algebra over $\mathbb{C}$, and let $Z(A)$ denote the center of A.

a) Suppose L is a finitely-generated irreducible A-module. Prove that any $z \in Z(A)$ acts on L by $\lambda \cdot \mathrm{id}_L$ for some scalar $\lambda \in \mathbb{C}$. (We refer to λ as the eigenvalue of z on L.)

b) Pick $z \in Z(A)$. Let M and N be A-modules for which z acts on M by $\lambda \cdot \mathrm{id}_M$ and z acts on N by $\mu \cdot \mathrm{id}_N$, for $\lambda, \mu \in \mathbb{C}$. Prove that when $\lambda \neq \mu$, any short exact sequence of A-modules of the form

$$0 \to M \xrightarrow{f} X \xrightarrow{g} N \to 0 \tag{5}$$

actually splits.

c) Continuing the above, find an explicit example where $\lambda = \mu$ and (5) does not split.

d) When A is semisimple, prove that for any two non-isomorphic irreducible representations L and L', there is an element $z \in Z(A)$ which has distinct eigenvalues on L and on L'.

e) Find an example of a finite-dimensional algebra A over $\mathbb{C}$, and two non-isomorphic irreducible modules L and L', such that for any $z \in Z(A)$ the eigenvalue of z on L and on L' agree. (In other words, the center can not distinguish between these irreducible modules.)

3. Let A be a finite-dimensional algebra over $\mathbb{C}$. For $a \in A$, let L_a denote left multiplication by a, a linear map $A \to A$. Define a bilinear pairing $A \times A \to \mathbb{C}$ which sends $(a, b) \mapsto \mathrm{trace}(L_a \circ L_b)$. Define the *radical* of A to be

$$\mathrm{rad}\, A := \{a \in A \mid (a, b) = 0 \text{ for all } b \in A\}. \tag{6}$$

a) Prove that $(-,-)$ is symmetric and associative. (A form on an algebra is *associative* if $(ab, c) = (a, bc)$.)

b) Prove that $\mathrm{rad}\, A$ is a two-sided ideal.

c) Let I be an ideal for which every element is nilpotent. Prove that $I \subset \mathrm{rad}\, A$. Also prove that I is contained in the Jacobson radical of A.

d) Prove that every element of $\mathrm{rad}\, A$ is nilpotent.

Note: There is a theorem which states that in a finite-dimensional algebra, the Jacobson radical is a nilpotent ideal. Hence $\mathrm{rad}\, A = J(A)$.

4. This exercise focuses on the complex representation theory of the finite group $G = GL_2(\mathbb{F}_3)$. Here is its character table: for each conjugacy class we give a representative matrix, and indicate the order of that matrix. Here, I represents the identity matrix.

C	I	$-I$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
$\lvert C \rvert$	1	1	α	8	8	6	β	γ
order	1	2	2	3	6	4	8	8
$\mathbb{1}$	1	1	1	1	1	1	1	1
det	1	1	-1	1	1	1	-1	-1
X	3	3	1	0	0	-1	-1	-1
Y	3	3	-1					
Z_1								
Z_2								
Z_3								
W								

You may freely use the fact that

$$(*): \quad \text{if } A \text{ conjugates } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ to an upper triangular matrix, then } A \text{ itself is upper triangular.}$$

a) Compute the size of the centralizer of $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. What is the size α of the conjugacy class of σ?

b) Fill in the column of the character table associated to σ.

c) Let $B \subset G$ be the subgroup of upper triangular matrices. Which conjugacy classes have non-empty intersection with B?

d) There is a one-dimensional representation L of B on which $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ acts by the scalar a. Let $W = \mathrm{Ind}_B^G L$. Use the formula for induced characters to compute the character of W.

e) Explain why W is irreducible.

f) Does L appear as a direct summand of the restricted representation $\mathrm{Res}_G^B X$ or $\mathrm{Res}_G^B Z_i$ for any i? Justify your answer.

Note: You do not need to fill in the entire character table.

5. Let B be a $\mathbb{C}$-algebra and $A \subset B$ be a subalgebra.

a) Restriction is a functor R from B-modules to A-modules. Describe R as tensor product with a bimodule. Describe R as Hom with a bimodule.

b) Induction is a functor I from A-modules to B-modules. We know that induction and restriction are adjoint functors (which is the left adjoint?). Construct explicitly the unit and counit of adjunction using bimodule morphisms.

c) Now let $A = \mathbb{C}[x^3] \subset B = \mathbb{C}[x]$. We aim to prove that induction and restriction are adjoint in the other direction as well (i.e. they are biadjoint). The unit of adjunction is the (B, B)-bimodule morphism given on the identity by

$$\Delta \colon B \to B \otimes_A B, \quad 1 \mapsto x^2 \otimes 1 + x \otimes x + 1 \otimes x^2.$$

Verify that Δ extends to a well-defined B-bimodule morphism.

d) Let $\partial \colon B \to A$ be the A-linear map defined by $\partial(1) = \partial(x) = 0$ and $\partial(x^2) = 1$. Check that ∂ is the counit of adjunction (paired with Δ) by confirming the adjunction relations.

e) Prove that the algebra of natural transformations from $R \circ I$ to itself is isomorphic to the algebra $\mathrm{Mat}_3(A)$ of 3×3 matrices with coefficients in A.

6. In this problem, we work over $\mathbb{C}$, and X is the algebraic set inside $\mathbb{A}^2$ cut out by the equation

$$x^2 - 4(y+1)^2 = 1.$$

a) There is a regular function $\varphi\colon X \to \mathbb{A}^1$ which comes from projecting X to the x-axis. Explicitly write down the corresponding homomorphism φ^* of $\mathbb{C}$-algebras, and justify your answer. Please use the t variable for $\mathbb{A}^1$, i.e. $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[t]$.

b) Give the definition of an integral extension of commutative rings. Is φ^* an integral extension? Prove your answer.

c) There is a family of homomorphisms $f_a^*\colon \mathbb{C}[t] \to \mathbb{C}[x,y]$ for $a \in \mathbb{C}$, which sends $t \mapsto x+ay$. Explicitly describe the corresponding regular function $f_a\colon \mathbb{A}^2 \to \mathbb{A}^1$.

d) For which a is f_a^* an integral extension? Prove your answer is correct.