1	2	3	4	5	6	7	8	9	10	Total

QUALIFYING EXAM, Fall 2020, Algebraic Topology

NAME ______________________________
(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ______________ **SIGNATURE** ______________

Please do all 10 problems.

Problem 1. (i) Define the Homotopy Lifting Property.

(ii) Let $n \geq 1$. Consider the map

$$g : S^{4n-2} \times S^5 \xrightarrow{\text{proj}} (S^{4n-2} \times S^5)/(S^{4n-2} \vee S^5) = S^{4n+3} \xrightarrow{\text{Hopf}} \mathbf{HP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups. Prove, however, that g is not homotopic to a constant map.

Problem 2. Let W be a compact manifold with boundary $\partial W = M$. Prove that the Euler characteristic $\chi(M)$ is even.

Problem 3. Prove that, if a CW-complex X is an Eilengberg MacLane space of type $K(\mathbf{Z}, n)$ for $n \geq 2$, then X has cells of arbitrarily high dimension.

Problem 4. State the Freudenthal Suspension Theorem. Prove that the Whitehead element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \to \pi_{n+k}(\Sigma(S^n \times S^k)).$$

Problem 5. Let $h : S^7 \to S^4$ be the Hopf map. Let $\lambda \geq 1$ be an integer, and the map $c_\lambda : S^7 \xrightarrow{\lambda} S^7 \vee ... \vee S^7$ divides the sphere S^7 into λ spheres. Define a map

$$f_\lambda : S^7 \xrightarrow{c_\lambda} S^7 \vee ... \vee S^7 \xrightarrow{h \vee ... \vee h} S^4.$$

Prove that the space $X_\lambda = S^4 \cup_{f_\lambda} D^8$ is homotopy equivalent to a closed compact manifold of dimension 8 if and only if $\lambda = 1$.

Problem 6. Let $f : S^{2n} \to S^n \times S^n$ be a map. Compute the reduced homology groups homomorphism: $f_* : \tilde{H}_*(S^{2n}) \to \tilde{H}_*(S^n \times S^n)$.

Problem 7. Let $X \subset S^n$ be homeomorphic to $S^p \vee S^q$, $1 \leq p < q \leq n-1$. Compute the homology groups $\widetilde{H}_q(S^n \setminus X)$.

Problem 8. Let M be a compact oriented closed manifold of dimension $\dim M = 2k$. Prove that if $H_{k-1}(M; \mathbf{Z})$ is torsion-free, then the group $H_k(M; \mathbf{Z})$ is also torsion-free.

Problem 9. Compute the homotopy group $\pi_q(S^2 \vee S^2)$ for $q = 1, 2, 3$.

Problem 10. Prove that any map $f : SO(3) \to T^4$ is contractible. Here $SO(3)$ is the special orthogonal group and $T^4 = S^1 \times S^1 \times S^1 \times S^1$ is a torus.