QUALIFYING EXAM, Fall 2020, Algebraic Topology

Problem 1. (i) Define the Homotopy Lifting Property.

(ii) Let $n \geq 1$. Consider the map

$$g : S^{4n-2} \times S^5 \xrightarrow{\text{proj}} (S^{4n-2} \times S^5)/(S^{4n-2} \vee S^5) = S^{4n+3} \xrightarrow{\text{Hopf}} \mathbf{HP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups. Prove, however, that g is not homotopic to a constant map.

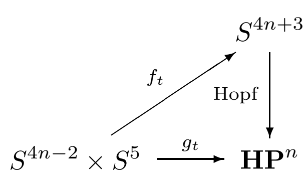
Solution. (ii) Consider the homomorphism $g_* : \pi_q(S^{4n-2} \times S^5) \xrightarrow{\text{proj}_*} \pi_q S^{4n+3} \xrightarrow{\text{Hopf}_*} \pi_q \mathbf{HP}^n$. Since $\pi_q(S^{4n-2} \times S^5) \cong \pi_q S^{4n-2} \oplus \pi_q S^5$, the homomorphism proj_* factors through the projections

$$\pi_q S^{4n+3} \leftarrow \pi_q S^{4n-2} \xleftarrow{\text{proj}_1} \pi_q(S^{4n-2} \times S^5) \xrightarrow{\text{proj}_2} \pi_q S^5 \to \pi_q S^{4n+3}$$

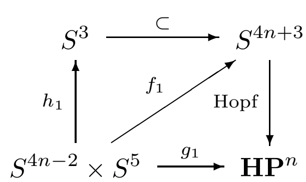
Since the maps $S^{4n-2} \to S^{4n+3}$ and $S^5 \to S^{4n+3}$ are homotopically trivial, the homomorphism $g_* : \pi_q(S^{4n-2} \times S^5) \to \pi_q S^{4n+3} \to \pi_q \mathbf{HP}^n$ is trivial as well.

The homomorphism $g_* : \tilde{H}_q(S^{4n-2} \times S^5) \xrightarrow{\text{proj}_*} \tilde{H}_q S^{4n+3} \xrightarrow{\text{Hopf}_*} \tilde{H}_q \mathbf{HP}^n$ is also trivial. Indeed, the group $\tilde{H}_{4n+3} S^{4n+3} = \mathbf{Z}$ and trivial ortherwise and $\tilde{H}_q \mathbf{HP}^n$ is non-trivial only if $q = 4, \ldots, 4n$, i.e., any homomorphism $\tilde{H}_q S^{4n+3} \to \tilde{H}_q \mathbf{HP}^n$ is trivial.

Now we assume that the map $g : S^{4n-2} \times S^5 \to \mathbf{HP}^n$ is contractible. Let $g_t : S^{4n-2} \times S^5 \to \mathbf{HP}^n$ be a homotopy such that $g_0 = g$ and $g_1(S^{4n-2} \times S^5) = x_0 \in \mathbf{HP}^n$. By the Lifting Homotopy Property, there exists a homotopy $f_t : S^{4n-2} \times S^5 \to S^{4n+3}$ such that the diagram



commutes. Here $f_0 = \text{proj}$ as above. Then we have that $f_1(S^{4n-2} \times S^5) \subset \text{Hopf}^{-1}(x_0) = S^3$. We notice that the homomorphism $\text{proj}_* : H_{4n+3}(S^{4n-2} \times S^5) \to H_{4n+3}(S^{4n+3})$ is non-trivial (it is an isomorphism). However $(f_1)_*$ must be trivial since the map f_1 factors through S^3:



Since $(f_1)_* = \text{proj}_*$, we obtain a contradiction. □

Problem 2. Let W be a compact manifold with boundary $\partial W = M$. Prove that the Euler characteristic $\chi(M)$ is even.

Solution. We use $\mathbf{Z}/2$ coefficients so that Poincaré duality applies. If W is even-dimensional, then M is an odd-dimensional manifold (say M has dimension n). Then $\text{rk}\, H_k(M; \mathbf{Z}/2) = \text{rk}\, H^{n-k}(M; \mathbf{Z}/2) = \text{rk}\, H_{n-k}(M; \mathbf{Z}/2)$ where the first equality comes from Poincaré duality and the second from the universal coefficient theorem and the fact we are working in field coefficients. These terms occur with opposite signs in the computation of Euler characteristic in $\mathbf{Z}/2$ coefficients, so $\chi(M) = 0$ and is therefore even.

(The fact that Euler characteristic is the same when using the rank of homology groups in F coefficients for any field F can be seen via counting terms using the Universal Coefficient Theorem.)

If instead W is odd-dimensional, then we form the double manifold $\tilde{W}$ by taking two copies of W, $W^{(1)}$ and $W^{(2)}$ and gluing them along their boundary so that $\tilde{W} = W^{(1)} \cup_{\partial W} W^{(2)}$. By the Mayer-Vietoris long exact sequence,

$$\chi(\tilde{W}) = \chi(W^{(1)}) + \chi(W^{(2)}) - \chi(M)$$

As $\tilde{W}$ is an odd-dimensional closed manifold its Euler characteristic is zero for reasons described above. Thus $\chi(M) = 2\chi(W)$ and is even. □

Problem 3. Prove that, if a CW-complex X is an Eilengberg MacLane space of type $K(\mathbf{Z}, n)$ for $n \geq 2$, then X has cells of arbitrarily high dimension.

Solution. It is sufficient to show that for any positive integer N, there is some $q > N$ such that $H^q(K(\mathbf{Z}, n)) \neq 0$ in some coefficients. We use the following

Theorem 1. *There is a one-to-one correspondence between cohomology operations of type* $(n, \pi; n', \pi')$ *and elements of* $H^{n'}(K(\pi, n); \pi')$.

Recall that $H^{2n}(\mathbb{CP}^\infty; \mathbf{Z}) = \mathbf{Z}[x]$, where $x \in H^2(\mathbb{CP}^\infty; \mathbf{Z})$. Let $\alpha = x^n$ be an element of $H^{2n}(\mathbb{CP}^\infty; \mathbf{Z})$. Then we know $\alpha \mapsto \alpha^k$ in $H^{2nk}(\mathbb{CP}^\infty; \mathbf{Z})$ is nontrivial in $H^*(\mathbb{CP}^\infty; \mathbf{Z})$. Therefore, $\alpha \mapsto \alpha^k$ is a nontrivial cohomology operation of type $(2n, \mathbf{Z}; 2kn, \mathbf{Z})$. Thus there is a nontrivial element of $H^{2nk}(K(\mathbf{Z}, n); \mathbf{Z})$. Since this can be done for any positive integer k, the space $K(\mathbf{Z}, n)$ has nontrivial cohomology groups in $\mathbf{Z}$ coefficients, in arbitrarily large dimensions, as desired. □

Problem 4. State the Freudenthal Suspension Theorem. Prove that the Whitehead element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \to \pi_{n+k}(\Sigma(S^n \times S^k)).$$

Solution. Let X be a space with a base point x_0. We construct the homomorphism

$$\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X) \tag{1}$$

as follows. Let $\alpha \in \pi_q(X)$, and a map $f : S^q \to X$ be a representative of α. The map

$$\Sigma f : \Sigma S^q = S^{q+1} \to \Sigma X$$

defined by the formula $\Sigma f(y,t) = (f(y), t) \in \Sigma X$ gives a representative for $\Sigma(a) \in \pi_{q+1}(\Sigma X)$. It is not hard to check that

1. $f \sim g$ implies that $\Sigma f \sim \Sigma g$;
2. $\Sigma f + \Sigma g \sim \Sigma(f+g)$.

The homomorphism Σ is the *suspension homomorphism.*

Theorem 1. (Freudenthal Theorem) *The suspension homomorphism*

$$\Sigma : \pi_q(S^n) \longrightarrow \pi_{q+1}(S^{n+1})$$

is isomorphism for $q \leq 2n-1$ *and epimorphism for* $q = 2n-1$.

Now consider the suspension homomorphism $\Sigma : \pi_q(S^n \vee S^k) \to \pi_{q+1}(\Sigma(S^n \vee S^k))$. First, we prove

Lemma 2. *The element* $w \in \pi_{n+k-1}(S^n \vee S^k)$ *is in a kernel of each of the following homomorphisms:*

(1) $i_* : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k-1}(S^n \times S^k)$,

(2) $pr_*^{(n)} : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k-1}(S^n)$, $pr_*^{(k)} : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k-1}(S^k)$.

Proof. The exact sequence

$$\to \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \xrightarrow{\partial} \pi_{n+k-1}(S^n \vee S^k) \xrightarrow{i_*} \pi_{n+k-1}(S^n \times S^k) \to$$

implies that $w \in \text{Ker } i_*$ since $w = \partial(\iota)$. The commutative diagram

$$\begin{array}{ccc} \pi_{n+k-1}(S^n \vee S^k) & \xrightarrow{i_*} & \pi_{n+k-1}(S^n \times S^k) \\ & \searrow^{pr_*^{(n)}} & \downarrow pr_* \\ & & \pi_{n+k-1}(S^n) \end{array}$$

(where $pr : S^n \times S^k \to S^n$ is a map collapsing S^k to the base point) implies that $w \in \text{Ker } pr_*^{(n)}$ and similarly $w \in \text{Ker } pr_*^{(n)}$. $\square$

Now we prove:

Lemma 3. *The element* $w \in \pi_{n+k-1}(S^n \vee S^k)$ *is in the kernel of the suspension homomorphism*

$$\Sigma : \pi_{n+k-1}(S^n \vee S^k) \to \pi_{n+k}(\Sigma(S^n \vee S^k)).$$

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} \pi_{n+k-1}(S^n) & \xleftarrow{pr_*^{(n)}} & \pi_{n+k-1}(S^n \vee S^k) & \xrightarrow{pr_*^{(k)}} & \pi_{n+k-1}(S^k) \\ \Sigma \downarrow & & \downarrow \Sigma & & \Sigma \downarrow \\ \pi_{n+k}(S^{n+1}) & \xleftarrow{\Sigma(pr_*^{(n)})} & \pi_{n+k}(\Sigma(S^n \vee S^k)) & \xrightarrow{\Sigma(pr_*^{(k)})} & \pi_{n+k}(S^{k+1}) \end{array} \qquad (2)$$

where pr denote the collapsing maps. By Lemma 2 $w \in \text{Ker } pr_*^{(n)}$, $w \in \text{Ker } pr_*^{(k)}$. Notice that $\Sigma(S^n \vee S^k) \sim S^{n+1} \vee S^{k+1}$. We need the following fact:

Lemma 4. *There is an isomorphism*

$$\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})$$

Proof. Consider the long exact sequence for the pair $(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1})$:

$$\begin{aligned} \pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \xrightarrow{\partial} \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{i_*} \pi_{n+k}(S^{n+1} \times S^{k+1}) \\ \xrightarrow{j_*} \pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \to \end{aligned} \qquad (3)$$

We notice that the $(n+k+1)$-skeleton of the product $S^{n+1} \times S^{k+1}$ is the wedge $S^{n+1} \vee S^{k+1}$. Thus any map $D^{k+n+1} \to S^{n+1} \times S^{k+1}$ may be deformed to the subcomplex $S^{n+1} \vee S^{k+1}$. Thus $\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0$. The same argument gives that

$$\pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0.$$

Thus the long exact sequence (3) gives the isomorphism:

$$i_* : \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{\cong} \pi_{n+k}(S^{n+1} \times S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1}). \quad \square$$

To complete the proof of Lemma 3 we notice that Lemma 4 and the diagram (2) imply that $w \in \text{Ker } \Sigma$. This proves Lemma 3. $\square$

Problem 5. Let $h : S^7 \to S^4$ be the Hopf map. Let $\lambda \geq 1$ be an integer, and the map $c_\lambda : S^7 \xrightarrow{\lambda} S^7 \vee ... \vee S^7$ divides the sphere S^7 into λ spheres. Define a map

$$f_\lambda : S^7 \xrightarrow{c_\lambda} S^7 \vee ... \vee S^7 \xrightarrow{h \vee ... \vee h} S^4.$$

Prove that the space $X_\lambda = S^4 \cup_{f_\lambda} D^8$ is homotopy equivalent to a closed compact manifold of dimension 8 if and only if $\lambda = 1$.

Solution. Note that if $\lambda = 1$ then $X_\lambda \cong \mathbb{HP}^2$, which is a closed compact 8-manifold.

For the other direction, we take a quick detour and compute $H(f)$, the Hopf invariant of f, for all values of λ. Indeed, X_λ has a cellular structure with one 0-cell, one 4-cell and one 8-cell; it must therefore have cohomology (and homology) groups isomorphic to $\mathbf{Z}$ in dimensions $0, 4$, and 8, and zero otherwise.

Recall that the Hopf invariant $H(h) = 1$, and that $[f] = \lambda[h] \in \pi_7(S^4)$ by construction of f. Since the Hopf invariant is additive, $H(f) = \lambda$. In particular, now applying the definition of Hopf invariant, let $a \in H^4(X_\lambda; \mathbf{Z})$, $d \in H^8(X_\lambda; \mathbf{Z})$ be generators. We conclude $a \cup a = \lambda d$.

Now, suppose that X_λ were homotopy equivalent to a closed 8-manifold M; homotopy equivalence preserves cup product structure. Then, as X is simply-connected, M must be also, and thus orientable. By Poincaré duality, there would be some $b \in H^2(M; \mathbf{Z})$ such that $a \cup b = d$. Note that since $H^4(M; \mathbf{Z}) \cong \mathbf{Z}$, that $b = \mu a$ for some $\mu \in \mathbf{Z}$. Then, $d = a \cup b = a \cup \mu a = \mu(a \cup a) = \mu\lambda d$. In order for $\mu\lambda = 1$ in $\mathbf{Z}$, we would need $|\mu| = |\lambda| = 1$. Thus, if $X_\lambda \simeq M$ is a closed 8-manifold, then $\lambda = 1$. □

Problem 6. Let $f : S^{2n} \to S^n \times S^n$ be a map. Compute the reduced homology groups homomorphism: $f_* : \tilde{H}_*(S^{2n}) \to \tilde{H}_*(S^n \times S^n)$.

Solution. We first show the induced map in cohomology is trivial, and then deduce that the induced map on homology must also be trivial.

Recall that the cup product structure of S^n is $\mathbf{Z}[\alpha]/\alpha^2$ with α in degree n, so $H^*(S^n \times S^n) = \mathbf{Z}[\alpha_1, \alpha_2]/(\alpha_1^2, \alpha_2^2)$. Also we have that $H^*(S^{2n}) = \mathbf{Z}[\beta]/\beta^2$, where β has degree $2n$.

Consider $f^* : H^*(S^n \times S^n) \to H^*(S^{2n})$. We are interested in $f^*(\alpha_1 \otimes \alpha_2)$, since $\alpha_1 \otimes \alpha_2$ generates $H^{2n}(S^n \times S^n)$. Since $H^n(S^{2n}) = 0$, we know that $f^*(\alpha \otimes 1) = f^*(1 \otimes \alpha) = 0$. By the naturality of cup product, we see that

$$\begin{aligned} f^*((\alpha \otimes \alpha)) &= f^*((\alpha \otimes 1) \smile (1 \otimes \alpha)) \\ &= f^*(\alpha \otimes 1) \smile f^*(1 \otimes \alpha) \\ &= 0 \smile 0 \end{aligned}$$

Therefore $f^* : H^{2n}(S^n \times S^n) \to H^{2n}(S^{2n})$ is the zero homomorphism.

To see that this means $f_* : H^{2n}(S^{2n}) \to H^{2n}(S^n \times S^n)$ is the zero homomorphism as well, we use the natural isomorphism arising from the universal coefficient theorem (since our Ext terms are zero):

$$H^{2n}(S^{2n}) \cong \mathrm{Hom}(H_{2n}(S^{2n}), \mathbf{Z})$$

The naturality of this isomorphism gives rise to the following commutative diagram:

$$\begin{array}{ccc} H^{2n}(S^{2n}) & \xrightarrow{f^*} & H^{2n}(S^n \times S^n) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(H_{2n}(S^{2n}), \mathbf{Z}) & \xrightarrow{(f_*)^*} & \mathrm{Hom}(H_{2n}(S^n \times S^n), \mathbf{Z}) \end{array}$$

As f^* is trivial, so must be $(f_*)^*$ and therefore f_*. □

Problem 7. Let $X \subset S^n$ be homeomorphic to $S^p \vee S^q$, $1 \leq p < q \leq n-1$. Compute the homology groups $\widetilde{H}_q(S^n \setminus X)$.

Solution. We need the following

Lemma 1. $\tilde{H}_q(S^n \setminus S^k) \cong \mathbf{Z}$ *exactly where* $q = n-k-1$ *and is zero otherwise.*

Let $X = S^n \setminus (S^p \vee S^q)$. To compute the homology groups $H_r(X)$, we will apply Mayer-Vietoris. Let $X_1 = S^n \setminus S^p$ and $X_2 = S^n \setminus S^q$. Then $X = X_1 \cap X_2$, and as both X_1 and X_2 are open (as they are the complement of the image of a compact set under continuous map) the Mayer-Vietoris theorem applies. Additionally,

$$X_1 \cup X_2 = S^n \setminus * \cong \mathbf{R}^n.$$

Now, consider the Mayer-Vietoris sequence in reduced homology groups (to avoid too many cases).

$$\to \tilde{H}_{r+1}(\mathbf{R}^n) \to \tilde{H}_r(X) \to \tilde{H}_r(S^n \setminus S^p) \oplus \tilde{H}_r(S^n \setminus S^q) \to \tilde{H}_r(\mathbf{R}^n) \to .$$

For any r, we have $\tilde{H}_r(\mathbf{R}^n) = 0$, so $\tilde{H}_r(X) \cong \tilde{H}_r(S^n \setminus S^p) \oplus \tilde{H}_r(S^n \setminus S^q)$. By Lemma 1, $\tilde{H}_q(S^n \setminus S^k) \cong \mathbf{Z}$ exactly where $q = n-k-1$ and is zero otherwise. Since $p < q$, we have

$$\tilde{H}_r(X) \cong \tilde{H}_r(S^n \setminus S^p) \oplus \tilde{H}_r(S^n \setminus S^q) = \begin{cases} \mathbf{Z} & r = n-p-1 \\ \mathbf{Z} & r = n-q-1 \\ 0 & \text{else.} \end{cases}$$

[In the case $p = q$, we have instead

$$\tilde{H}_r(X) \cong \tilde{H}_r(S^n \setminus S^p) \oplus \tilde{H}_r(S^n \setminus S^q) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & r = n-p-1 \\ 0 & \text{else.} \end{cases}$$

The only difference between reduced and nonreduced homology is a $\mathbf{Z}$ summand in $H_0(X)$, so this completes the computation.] □

Problem 8. Let M be a compact oriented closed manifold of dimension $\dim M = 2k$. Prove that if $H_{k-1}(M; \mathbf{Z})$ is torsion-free, then the group $H_k(M; \mathbf{Z})$ is also torsion-free.

Solution. By the Universal Coefficient Theorem,

$$H^k(X; \mathbf{Z}) \cong \mathrm{Hom}(H_k(X), \mathbf{Z}) \oplus \mathrm{Ext}(H_{k-1}(X); \mathbf{Z}).$$

As X is a closed manifold, it has the homotopy type of a finite CW complex, so $H_{k-1}(X; \mathbf{Z})$ must be finitely generated. Thus $\mathrm{Ext}(H_{k-1}(X; \mathbf{Z}))$ is equal to $\mathrm{Ext}(T(H_{k-1}(X; \mathbf{Z})))$, which is trivial. Thus $H^k(X; \mathbf{Z}) \cong \mathrm{Hom}(H_k(X), \mathbf{Z})$. But this too must be torsion-free, since $H_k(X)$ is finitely generated.

Lastly, by Poincaré duality, $H_k(X; \mathbf{Z}) \cong H^k(X; \mathbf{Z})$ as groups; therefore $H_k(X; \mathbf{Z})$ is torsion-free. □

Problem 9. Compute the homotopy group $\pi_q(S^2 \vee S^2)$ for $q = 1, 2, 3$.

Solution. We have that $\pi_1(S^2 \vee S^2) = 0$ since $S^2 \vee S^2$ has a CW-decomposition with one zero cell and two 2-cells. Then the CAT iplies that any map $S^1 \to S^2 \vee S^2$ is homotopic to a constant map. Then we know that $\pi_2(S^2 \times S^2) \cong \pi_2(S^2) \oplus \pi_2(S^2) = \mathbf{Z} \oplus \mathbf{Z}$. We notice that the inclusion map $i : S^2 \vee S^2 \to S^2 \times S^2$ induces an isomorphism $\pi_2(S^2 \vee S^2) \to \pi_2(S^2 \times S^2)$. Indeed, we have that

$$S^2 \times S^2 = (S^2 \vee S^2) \cup_w D^4,$$

where $w : S^3 \to S^2 \vee S^2$ is the Whitehead map.

Now we have that the homomorphism $i_* : \pi_3(S^2 \vee S^2) \to \pi_3(S^2 \times S^2)$ has the kernel given by the attaching map $w : S^3 \to S^2 \vee S^2$. Thus we have the exact sequence

$$\mathbf{Z} \to \pi_3(S^2 \vee S^2) \to \pi_3(S^2 \times S^2)$$

Thus we conclude $\pi_3(S^2 \vee S^2) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. □

Problem 10. Prove that any map $f : SO(3) \to T^4$ is contractible. Here $SO(3)$ is the special orthogonal group and $T^4 = S^1 \times S^1 \times S^1 \times S^1$ is a torus.

Solution. First, we recall that $SO(3)$ is homeomorphic to $\mathbb{RP}^3$. We consider the universal covering $p : \mathbf{R}^4 \to T^4$. Let $f : \mathbb{RP}^3 \to T^4$ be a map, where a base point $x_0 \in \mathbb{RP}^3$ maps to $y_0 = f(x_0)$. Let $\tilde{y}_0 \in p^{-1}(y_0)$ be a point in $\mathbf{R}^4$. We know that $\pi_1\mathbb{RP}^3 = \mathbf{Z}_2$, and $\pi_1 T^4 = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. Thus $f_*(\pi_1\mathbb{RP}^3) = 0 \in \pi_1 T^4$ for any map $f : \mathbb{RP}^3 \to T^4$. We use the following result:

Theorem 5. *Let $p : T \to X$ be a covering space, and Z be a path-connected space, $x_0 \in X$, $\tilde{x}_0 \in T$, $p(\tilde{x}_0) = x_0$. Given a map $f : (Z, z_0) \to (X, x_0)$ there exists a lifting $\tilde{f} : (Z, z_0) \to (T, \tilde{x}_0)$ if and only if $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \tilde{x}_0))$.*

In our case, it means that there exists a map $\tilde{f} : \mathbf{RP}^3 \to \mathbf{R}^4$ such that $\tilde{f}(x_0) = \tilde{y}_0$ and $\tilde{f} = p \circ f$. Thus the homotopy class of f is in the image of $p_* : \pi_1\mathbf{R}^4 \to T^4$. Hence $f \sim 0$. □