ALGEBRA QUALIFYING EXAMINATION, FALL 2010

Your Name:

Part I. Carefully state each of the following (6 points each):

1. Wedderburn-Artin Theorem.
2. Lying Over, Going Up, Incomparability, and Maximality Theorem.
3. Three equivalent definitions of dimension of an irreducible affine algebraic set.

Part II. True or false? If true provide a brief explanation, if false provide a counterexample (6 points each):

1. $\mathrm{Aut}(C_8) \cong C_4$.
2. Let R be a commutative ring and I, J be two ideals of R. If the modules R/I and R/J are isomorphic then $I = J$.
3. If R is an integral domain and $r \in R$ is an irreducible element of R, then (r) is a prime ideal of R.
4. If R is a commutative ring and V is an R-module, then $V = 0$ if and only if $V_M = 0$ for every maximal ideal M of R.
5. $\{(x, y) \in \mathbb{A}^2 \mid x^2 + y^2 = 1\}$ is homeomorphic to $\mathbb{A}^1$ in Zariski topology.

Part III. Attempt any four of the following five problems (13 points each; you can only get credit for four):

1. Let $\varphi : G \to H$ be a surjective homomorphism of finite groups. If P is a Sylow p-subgroup of G, then $\varphi(P)$ is a Sylow p-subgroup of H. Conversely, every Sylow p-subgroup of H is the image of a certain Sylow p-subgroup of G.
2. Let F be a field and $A \in M_n(F)$ be $n \times n$ matrix over F. Show that A is similar to its transpose A^t.
3. Let G be a finite group, V be a $\mathbb{C}G$-module and $\chi = \chi_V$ be the character of V.

   (a) Explain why the action of G on $V \otimes V$ induces the structure of a $\mathbb{C}G$-module on $S^2(V)$.

   (b) Show that $\chi_{S^2(V)}(g) = (\chi(g)^2 + \chi(g^2))/2$.
4. Let R, R' be rings and $\mathfrak{F} : R\text{-}\mathtt{Mod} \to R'\text{-}\mathtt{Mod}$ be a functor left adjoint to a functor $\mathfrak{G} : R'\text{-}\mathtt{Mod} \to R\text{-}\mathtt{Mod}$.

(a) If $0 \to U' \to V' \to W' \to 0$ is a short exact sequence of R'-modules, then $0 \to \mathfrak{G}U' \to \mathfrak{G}V' \to \mathfrak{G}W'$ is an exact sequence of R-modules.

(b) If $\mathfrak{G}$ is exact and P is a projective R-module, then $\mathfrak{F}P$ is a projective R'-module.

5. Every element in a finite field can be written as a sum of two squares.