F.Q1. Consider the map
\[ f : \mathbb{C}^2 \to \mathbb{R}^2, \quad f(z, w) = (|z|^2 + |w|^2, 2|z|^2 - 3|w|^2). \]
Prove that pre-image \( f^{-1}(1, 0) \subset \mathbb{C}^2 \) is a smooth, compact, 2-dimensional manifold.

F.Q2. Let \( \mathbb{RP}^2 \equiv S^2/\sim \) be the real projective space. Let \( f : \mathbb{RP}^2 \to \mathbb{R}^3 \) be a map defined by
\[ f([x, y, z]) = \frac{1}{x^2 + y^2 + z^2} (yz, zx, xy). \]
(F.Q2a) Show that \( f \) is smooth.
(F.Q2b) Find a point in \( \mathbb{RP}^2 \) to show that \( f \) is not an immersion.

F.Q3. Let \( F : S^3 \to S^2 \) be a smooth map between spheres.
(F.Q3a) Show that there exist a smooth 2-form \( \omega \) on \( S^2 \) such that \( \int_{S^2} \omega = 1 \), and a smooth 1-form \( \eta \) on \( S^3 \) such that \( F^* \omega = d\eta \).
(F.Q3b) Let \( \tilde{\omega} \) be another smooth 2-form on \( S^2 \) satisfying \( \int_{S^2} \tilde{\omega} = 1 \). Show that there is a smooth 1-form \( \tau \) on \( S^2 \) such that \( \tilde{\eta} \equiv \eta + F^* \tau \) satisfies \( F^* \tilde{\omega} = d\tilde{\eta} \).
(F.Q3c) Show that
\[ \int_{S^3} \eta \wedge d\eta = \int_{S^3} \tilde{\eta} \wedge d\tilde{\eta}. \]

F.Q4. Let \( M^3 \to \mathbb{R}^3 \) be a compact, connected, 3-dimensional smooth submanifold/domain with boundary \( \partial M \) which has the induced (Stokes) orientation. For any point \( p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3 \), define the translated submanifold
\[ M_{p_0} \triangleq \{(x + x_0, y + y_0, z + z_0), (x, y, z) \in M\}. \]
Compute the limit
\[ \lim_{p_0 \to \infty} \int_{\partial M_{p_0}} \omega \]
where \( \omega \) is the 2-form
\[ \omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}. \]

F.Q5. Consider two Riemannian metrics \( g \) and \( u^2 \cdot g \) on some smooth manifold \( M^n \), where \( u \) is a smooth positive function on \( M \). Let \( (U, \varphi, x) \) be a local coordinate chart on \( M \). Compute the Christoffel symbols \( \Gamma(u^2 \cdot g)^k_{ij} \) of Riemannian connection associated to metric \( u^2 \cdot g \) in terms of \( u \) (along with its derivatives) and the Christoffel symbols \( \Gamma^k_{ij} = \Gamma(g)^k_{ij} \).
F.Q6. Consider parametrized surface \( \Sigma^2 \subset \mathbb{R}^3 \) defined by
\[
x = 3u - u^3 + 3uv^2, \quad y = -3v + v^3 - 3u^2v, \quad z = 3u^2 - 3v^2,
\]
where \((u, v) \in \mathbb{R}^2\). Show that its first fundamental form \( I \) (induced metric) is rotationally symmetric. I.e., there is a change of coordinates \((u, v) \to (w, \theta)\) such that
\[
I = h(w)^2 dw^2 + k(w)^2 d\theta^2,
\]
where \(h(w)\) and \(k(w)\) are two positive functions of \(w\).

F.Q7. Suppose the metric on some surface \( \Sigma^2 \) is given by
\[
ds^2 = du^2 + 2 \cos f(u, v) dudv + dv^2,
\]
where \(f\) is a function of \((u, v)\) taking value in \((-1, 1)\). Show that the Gauss curvature \(K = -\frac{\partial^2 f}{\sin f}\).

F.Q8. Let \((M^n, g)\) be a complete Riemannian manifold. Let \(N^k\) and \(W^l\) be two embedded submanifolds of \(M\) with no boundary, \(1 \leq k, l \leq n - 1\). Assume (i) \(N\) is compact, (ii) \(W\) is a closed subset (not necessarily compact), and (iii) \(N \cap W = \emptyset\).

(F.Q8a) Prove that there exists a minimal geodesic \(\gamma : [0, 1] \to M\) with \(\gamma(0) \in N\) and \(\gamma(1) \in W\) such that length \(L(\gamma) = \inf_{p \in N \text{ and } q \in W} d_M(p, q)\), where \(d_M\) is the distance function on \(M\) induced from metric \(g\).

(F.Q8b) Prove that the geodesic \(\gamma\) in part (F.Q8a) is perpendicular to \(N\) at \(t = 0\), and to \(W\) at \(t = 1\).

F.Q9. Consider torus \(T^3 = S^1 \times S^1 \times S^1\) with \(S^1\) treated as \([-\pi, \pi]\) with \(-\pi\) and \(\pi\) identified.

(F.Q9a) Does it admit a metric with positive Ricci curvature everywhere?

(F.Q9b) If not above, construct (not-necessarily in a closed formula) a smooth metric \(g\) on \(T^3\) such that all the sectional curvature is \(+1\) on the following sub-domain
\[
(-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \subset T^3.
\]

F.Q10. Let \(G^n\) be a Lie group of dimension \(n\), with identity element \(e\). Recall that multiplication from the left by an element \(h \in G\) defines a diffeomorphism \(L_h : G \to G, L_h(g) = h \cdot g\). A differential form \(\tilde{\omega}\) on \(G\) is said to be left invariant if it satisfies the condition \((L_h)^* \tilde{\omega} = \tilde{\omega}\) for all \(h \in G\).

(F.Q10a) Prove that any covector \(v^* \in T^*_e G\) uniquely extends to a smooth, left-invariant 1-form \(\omega\) on \(G\).

(F.Q10b) Use the result of (F.Q10a) to prove that there exist \(n\) pointwise independent, left-invariant 1-forms \(\omega^k\) on \(G\), \(k = 1, 2, \cdots, n\).

(F.Q10c) Prove that there exist constants \(c_{ij}^k\), such that for each \(\omega^k\) in (F.Q10b)
\[
d\omega^k = \sum_{1 \leq i < j \leq n} c_{ij}^k \cdot \omega^i \wedge \omega^j.
\]