Qualifying Exam, Differential Geometry
Fall 2020

F.Q1. Consider the map

$$f : \mathbb{C}^2 \to \mathbb{R}^2, \qquad f(z, w) = (|z|^2 + |w|^2, 2|z|^2 - 3|w|^2).$$

Prove that pre-image $f^{-1}(1, 0) \subset \mathbb{C}^2$ is a smooth, compact, 2-dimensional manifold.

F.Q2. Let $\mathbb{R}\mathrm{P}^2 \equiv S^2/\sim$ be the real projective space. Let $f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3$ be a map defined by

$$f([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(yz, zx, xy).$$

(F.Q2a) Show that f is smooth.

(F.Q2b) Find a point in $\mathbb{R}\mathrm{P}^2$ to show that f is not an immersion.

F.Q3. Let $F : S^3 \to S^2$ be a smooth map between spheres.

(F.Q3a) Show that there exist a smooth 2-form ω on S^2 such that $\int_{S^2} \omega = 1$, and a smooth 1-form η on S^3 such that $F^*\omega = d\eta$.

(F.Q3b) Let $\tilde{\omega}$ be another smooth 2-form on S^2 satisfying $\int_{S^2} \tilde{\omega} = 1$. Show that there is a smooth 1-form τ on S^2 such that $\tilde{\eta} \doteqdot \eta + F^*\tau$ satisfies $F^*\tilde{\omega} = d\tilde{\eta}$.

(F.Q3c) Show that

$$\int_{S^3} \eta \wedge d\eta = \int_{S^3} \tilde{\eta} \wedge d\tilde{\eta}.$$

F.Q4. Let $M^3 \to \mathbb{R}^3$ be a compact, connected, 3-dimensional smooth submanifold/domain with boundary ∂M which has the induced (Stokes) orientation. For any point $p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, define the translated submanifold

$$M_{p_0} \doteqdot \{(x + x_0, y + y_0, z + z_0),\ (x, y, z) \in M\}.$$

Compute the limit

$$\lim_{p_0 \to \infty} \int_{\partial M_{p_0}} \omega$$

where ω is the 2-form

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

F.Q5. Consider two Riemannian metrics g and $u^2 \cdot g$ on some smooth manifold M^n, where u is a smooth positive function on M. Let (U, φ, x) be a local coordinate chart on M. Compute the Christoffel symbols $\Gamma(u^2 \cdot g)_{ij}^k$ of Riemannian connection associated to metric $u^2 \cdot g$ in terms of u (along with its derivatives) and the Christoffel symbols $\Gamma_{ij}^k = \Gamma(g)_{ij}^k$.

F.Q6. Consider parametrized surface $\Sigma^2 \subset \mathbb{R}^3$ defined by

$$x = 3u - u^3 + 3uv^2, \quad y = -3v + v^3 - 3u^2 v, \quad z = 3u^2 - 3v^2,$$

where $(u, v) \in \mathbb{R}^2$. Show that its first fundamental form I (induced metric) is rotationally symmetric. I.e., there is a change of coordinates $(u, v) \to (w, \theta)$ such that

$$I = h(w)^2 dw^2 + k(w)^2 d\theta^2,$$

where $h(w)$ and $k(w)$ are two positive functions of w.

F.Q7. Suppose the metric on some surface Σ^2 is given by

$$ds^2 = du^2 + 2\cos f(u, v) du dv + dv^2,$$

where f is a function of (u, v) taking value in $(-1, 1)$. Show that the Gauss curvature $K = -\frac{\frac{\partial^2 f}{\partial u \partial v}}{\sin f}$.

F.Q8. Let (M^n, g) be a complete Riemannian manifold. Let N^k and W^l be two embeded submanifolds of M with no boundary, $1 \leq k, l \leq n - 1$. Assume (i) N is compact, (ii) W is a closed subset (not necessarily compact), and (iii) $N \cap W = \emptyset$.

(F.Q8a) Prove that there exists a minimal geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) \in N$ and $\gamma(1) \in W$ such that length

$$L(\gamma) = \inf_{p \in N \text{ and } q \in W} d_M(p, q),$$

where d_M is the distance function on M induced from metric g.

(F.Q8b) Prove that the geodesic γ in part (F.Q8a) is perpendicular to N at $t = 0$, and to W at $t = 1$.

F.Q9. Consider torus $T^3 = S^1 \times S^1 \times S^1$ with S^1 treated as $[-\pi, \pi]$ with $-\pi$ and π identified.

(F.Q9a) Does it admit a metric with positive Ricci curvature everywhere?

(F.Q9b) If not above, construct (not-necessarily in a closed formula) a smooth metric g on T^3 such that all the sectional curvature is $+1$ on the following sub-domain

$$(-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \subset T^3.$$

F.Q10. Let G^n be a Lie group of dimension n, with identity element e. Recall that multiplication from the left by an element $h \in G$ defines a diffeomorphism $L_h : G \to G$, $L_h(g) = h \cdot g$. A differential form $\tilde{\omega}$ on G is said to be **left invariant** if it satisfies the condition $(L_h)^* \tilde{\omega} = \tilde{\omega}$ for all $h \in G$.

(F.Q10a) Prove that any covector $v^* \in T_e^* G$ uniquely extends to a smooth, left-invariant 1-form ω on G.

(F.Q10b) Use the result of (F.Q10a) to prove that there exist n pointwise independent, left-invariant 1-forms ω^k on G, $k = 1, 2, \cdots, n$.

(F.Q10c) Prove that there exist constants c_{ij}^k, such that for each ω^k in (F.Q10b)

$$d\omega^k = \sum_{1 \leq i < j \leq n} c_{ij}^k \cdot \omega^i \wedge \omega^j.$$