F.Q1. Consider the map 
\[ f : \mathbb{C}^2 \to \mathbb{R}^2, \quad f(z, w) = (|z|^2 + |w|^2, 2|z|^2 - 3|w|^2). \]
Prove that pre-image \( f^{-1}(1, 0) \subset \mathbb{C}^2 \) is a smooth, compact, 2-dimensional manifold.

\[ f^{-1}(1, 0) \subset \mathbb{C}^2 \]

F.Q1 Sol. Use real coordinates \((x, y, u, v)\) where 
\[ z = x + iy \quad \text{and} \quad w = u + iv, \]
the Jacobian of map 
\[ f(x, y, u, v) = (x^2 + y^2 + u^2 + v^2, 2x^2 + 2y^2 - 3u^2 - 3v^2) \]
is given by 
\[ Df = \begin{bmatrix} 2x & 2y & 2u & 2v \\ 4x & 4y & -6u & -6v \end{bmatrix}. \]
If \( Df \) has rank \( \leq 1 \) on \( f^{-1}(1, 0) \) we would have 
\[ -20xu = 0, \quad -20xv = 0, \quad -20yu = 0, \quad -20yv = 0, \]
where the first equality is the determinant of the submatrix formed by the first and third columns. This implies \((x^2 + y^2)(u^2 + v^2) = 0\), which contradicts with \( x^2 + y^2 + u^2 + v^2 = 1 \). Hence \( Df \) has rank 2 on \( f^{-1}(1, 0) \), and by the transversality theorem we conclude that \( f^{-1}(1, 0) \) is a smooth 2-dimensional manifold. The compactness of \( f^{-1}(1, 0) \) follows from that \( f^{-1}(1, 0) \) is a subset of \( S^3(1) \).

F.Q2. Let \( \mathbb{RP}^2 \equiv S^2/\sim \) be the real projective space. Let \( f : \mathbb{RP}^2 \to \mathbb{R}^3 \) be a map defined by 
\[ f([x, y, z]) = \frac{1}{x^2 + y^2 + z^2} (yz, zx, xy). \]
(F.Q2a) Show that \( f \) is smooth.
(F.Q2b) Find a point in \( \mathbb{RP}^2 \) to show that \( f \) is not an immersion.

F.Q2 Sol. (F.Q2a) We may cover \( S^2 \) (and hence for \( \mathbb{RP}^2 \)) by \( S_{x,+} = \{(x, y, z), x > 0, x^2 + y^2 + z^2 = 1\} \) and 5 others similarly defined. Then \( \varphi_{x,+} : S_{x,+} \to (y, z) \) and 5 others defines a smooth coordinate atlas on \( \mathbb{RP}^2 \).

To see that \( f \) is smooth it is sufficient to check 
\[ f \circ \varphi_{x,+}^{-1} (y, z) = (yz, z\sqrt{1 - y^2 - z^2}, y\sqrt{1 - y^2 - z^2}) \]
is smooth on \( \{(y, z), y^2 + z^2 < 1\} \), and 5 others is smooth also. This is clearly true for \( f \circ \varphi_{x,+}^{-1} \) and for 5 others by symmetry.

(F.Q2b) At point \((y, z) = (0, 0)\) we compute the Jacobian of \( f \circ \varphi_{x,+}^{-1} \) and find it is a zero matrix. Hence \( f \) can not be an immersion.

F.Q3. Let \( F : S^3 \to S^2 \) be a smooth map between spheres.
(F.Q3a) Show that there exist a smooth 2-form \( \omega \) on \( S^2 \) such that \( \int_{S^2} \omega = 1 \), and a smooth 1-form \( \eta \) on \( S^3 \) such that \( F^*\omega = d\eta \).

(F.Q3b) Let \( \tilde{\omega} \) be another smooth 2-form on \( S^2 \) satisfying \( \int_{S^2} \tilde{\omega} = 1 \). Show that there is a smooth 1-form \( \tau \) on \( S^2 \) such that \( \tilde{\eta} \equiv \eta + F^*\tau \) satisfies \( F^*\tilde{\omega} = d\tilde{\eta} \).

(F.Q3c) Show that
\[
\int_{S^3} \eta \wedge d\eta = \int_{S^3} \tilde{\eta} \wedge d\tilde{\eta}.
\]

**F.Q3 Sol.** (F.Q3a) Choose local coordinate chart with \( S_{x,+} = \{(x, y, z), x > 0, x^2 + y^2 + z^2 = 1 \} \) and \( \varphi_{x,+} : S_{x,+} \to (y, z) \). Let \( \rho \) by a cutoff function supported on \( S_{x,+} \). Then \( \rho dy \wedge dz \) is a smooth 2-form on \( S^2 \) with \( \int_{S^2} \rho dy \wedge dz > 0 \). We may choose
\[
\omega = \frac{1}{\int_{S^2} \rho dy \wedge dz} \cdot \rho dy \wedge dz.
\]

Note that \( d\omega = 0 \).

Note that \( dF^*\omega = F^*d\omega = F^*0 = 0 \) and cohomology class \( [F^*\omega] \in H^2_{DeRh}(S^3) = 0 \), hence there is a smooth 1-form \( \eta \) on \( S^3 \) such that \( F^*\omega = d\eta \).

(F.Q3b) Since as cohomology class \( [\omega] = [\tilde{\omega}] \in H^2_{DeRh}(S^2) = \mathbb{R} \), there is a smooth 1-form \( \tau \) on \( S^2 \) such that \( \tilde{\omega} = \omega + d\tau \). Define \( \tilde{\eta} = \eta + F^*\tau \), we verify
\[
F^*\tilde{\omega} = F^*(\omega + d\tau) = d\eta + dF^*\tau = d\tilde{\eta}.
\]

(F.Q3c) Note that
\[
\eta \wedge dF^*\tau = -d(\eta \wedge F^*\tau) + d\eta \wedge F^*\tau
\]
we compute
\[
\int_{S^3} \tilde{\eta} \wedge d\tilde{\eta} - \int_{S^3} \eta \wedge d\eta
= \int_{S^3} \eta \wedge dF^*\tau + F^*\tau \wedge d\eta + F^*\tau \wedge dF^*\tau
= \int_{S^3} -d(\eta \wedge F^*\tau) + 2d\eta \wedge F^*\tau + F^*\tau \wedge dF^*\tau
= \int_{S^3} 2F^*\omega \wedge F^*\tau + F^*\tau \wedge dF^*\tau \quad \text{by Stokes theorem}
= \int_{S^3} F^*(2\omega \wedge \tau + \tau \wedge d\tau)
= 0,
\]
where the last equality is due to \( 0 = 2\omega \wedge \tau + \tau \wedge d\tau \in \Omega^3(S^2) \).

**F.Q4.** Let \( M^3 \to \mathbb{R}^3 \) be a compact, connected, 3-dimensional smooth submanifold/domain with boundary \( \partial M \) which has the induced (Stokes) orientation. For any point \( p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3 \), define the translated submanifold
\[
M_{p_0} \doteq \{(x + x_0, y + y_0, z + z_0), \ (x, y, z) \in M \}.
\]
Compute the limit
\[
\lim_{p_0 \to \infty} \int_{\partial M_{p_0}} \omega
\]
where \(\omega\) is the 2-form
\[
\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.
\]

**F.Q4 Sol.** First we show \(d\omega = 0\) by computing
\[
d\omega = \frac{dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}
- \frac{3(xdx + ydy + zdz)}{(x^2 + y^2 + z^2)^{5/2}} \wedge (xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)
= 0.
\]

Note that for \(p_0\) far away from origin \(\vec{0}\) we have \(\vec{0} \notin M_{p_0}\). Since \(\omega\) is smooth on \(\mathbb{R}^3 \setminus \{0\}\), we have
\[
\int_{\partial M_{p_0}} \omega = \int_{M_{p_0}} d\omega = 0
\]
by Stokes theorem. The limit follows. \(\square\)

**F.Q5.** Consider two Riemannian metrics \(g\) and \(u^2 \cdot g\) on some smooth manifold \(M^n\), where \(u\) is a smooth positive function on \(M\). Let \((U, \varphi, x)\) be a local coordinate chart on \(M\). Compute the Christoffel symbols \(\Gamma(u^2 \cdot g)^k_{ij}\) of Riemannian connection associated to metric \(u^2 \cdot g\) in terms of \(u\) (along with its derivatives) and the Christoffel symbols \(\Gamma^k_{ij} = \Gamma(g)^k_{ij}\).

**F.Q5 Sol.** We compute
\[
\Gamma(u^2 \cdot g)^k_{ij} = \frac{1}{2} (u^2 \cdot g)^k_{jl} (\partial_i (u^2 \cdot g)_{jl} + \partial_j (u^2 \cdot g)_{il} - \partial_l (u^2 \cdot g)_{ij})
= \frac{1}{2} u^{-2} \cdot g^{kl} (u^2 \partial_i g_{jl} + u^2 \partial_j g_{il} - u^2 \partial_l g_{ij} + 2uu_i g_{jl} + 2uu_j g_{il} - 2uu g_{ij})
= \Gamma(g)^k_{ij} + \frac{1}{u} \delta^k_i u_i + \frac{1}{u} \delta^k_j u_j - \frac{1}{u} g^{kl} g_{ij} u_i,
\]
where \(u_i = \partial_i u = \frac{\partial u}{\partial x^i}\). \(\square\)

**F.Q6.** Consider parametrized surface \(\Sigma^2 \subset \mathbb{R}^3\) defined by
\[
x = 3u - u^3 + 3uv^2, \quad y = -3v + v^3 - 3u^2 v, \quad z = 3u^2 - 3v^2,
\]
where \((u, v) \in \mathbb{R}^2\). Show that its first fundamental form \(I\) (induced metric) is rotationally symmetric. I.e., there is a change of coordinates \((u, v) \to (w, \theta)\) such that
\[
I = h(w)^2 dw^2 + k(w)^2 d\theta^2,
\]
where \( h(w) \) and \( k(w) \) are two positive functions of \( w \).

**F.Q6 Sol.** We compute

\[
I = (3 - 3a^2 + 3v^2)du + 6uvdv)^2 + (-6uvdu + (-3 + 3v^2 - 3a^2)dv^2 + (6udu - 6uvd^2)
\]

\[= 9(1 + a^2 + v^2)da^2 + 9(1 + u^2 + v^2)dv^2\]

Let \( u = w \cos \theta \) and \( v = w \sin \theta \), then

\[
I = 9(1 + w^2)(dw^2 + w^2d\theta^2),
\]

which is rotationally symmetric.

**F.Q7.** Suppose the metric on some surface \( \Sigma^2 \) is given by

\[
ds^2 = du^2 + 2 \cos f(u,v)du \cdot dv + dv^2,
\]

where \( f \) is a function of \((u,v)\) taking value in \((-1,1)\). Show that the Gauss curvature \( K = -\frac{\partial^2 f}{\sin f} \).

**F.Q7 Sol.** From the following formula for Gauss curvature

\[
K = \left| \begin{array}{ccc}
-\frac{1}{2}E_{uu} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u - \frac{1}{2}E_v & 0 \\
F_v - \frac{1}{2}G_u & E & F \\
\frac{1}{2}G_v & F & G
\end{array} \right| - \left| \begin{array}{ccc}
0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\
\frac{1}{2}E_v & E & F \\
\frac{1}{2}G_u & F & G
\end{array} \right|
\]

\[
(EG - F^2)^2
\]

we have

\[
K = \left| \begin{array}{ccc}
- \cos f \cdot f_u & - \sin f \cdot f_v & 0 \\
- \sin f \cdot f_v & \cos f & 0 \\
0 & \cos f & 1
\end{array} \right| - \left| \begin{array}{ccc}
0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\
\frac{1}{2}E_v & E & F \\
\frac{1}{2}G_u & F & G
\end{array} \right|
\]

\[
\left(1 - \cos^2 f\right)^2
\]

\[
(1 - \cos^2 f) \cdot f_{uv} + \sin f \cdot f_v \cdot \sin f \cdot f_u
\]

\[
\frac{\sin^4 f}{\sin^4 f}
\]

\[
\frac{f_{uv}}{\sin f}.
\]

**F.Q8.** Let \((M^n,g)\) be a complete Riemannian manifold. Let \( N^k \) and \( W^l \) be two embedded submanifolds of \( M \) with no boundary, \( 1 \leq k, l \leq n - 1 \). Assume (i) \( N \) is compact, (ii) \( W \) is a closed subset (not necessarily compact), and (iii) \( N \cap W = \emptyset \).

**F.Q8a** Prove that there exists a minimal geodesic \( \gamma : [0,1] \rightarrow M \) with \( \gamma(0) \in N \) and \( \gamma(1) \in W \) such that length

\[
L(\gamma) = \inf_{p \in N \text{ and } q \in W} d_M(p,q),
\]
where \( d_M \) is the distance function on \( M \) induced from metric \( g \).

(F.Q8b) Prove that the geodesic \( \gamma \) in part (F.Q8a) is perpendicular to \( N \) at \( t = 0 \), and to \( W \) at \( t = 1 \).

**F.Q8 Sol.** (F.Q8a) Let \( \{(p_i, q_i)\} \) be a sequence of point pairs in \( N \times W \) such that
\[
\inf_{p \in N \text{ and } q \in W} d_M(p, q) \quad \text{as } i \to \infty.
\]

Since \( N \) is compact, there is a subsequence (still indexed by \( i \)) such that \( p_i \to p_\infty \in N \). Hence the corresponding sequence \( \{q_i\} \) is in a bounded neighborhood of \( q_\infty \). Since \( M \) is complete, by taking another subsequence we may assume \( q_i \to q_\infty \). Since \( W \) is closed we have \( q_\infty \in W \), hence
\[
0 < d_M(p_\infty, q_\infty) = \inf_{p \in N \text{ and } q \in W} d_M(p, q)
\]
We may take \( \gamma : [0, 1] \to M \) to be a minimal geodesic joining \( p_\infty \) and \( q_\infty \).

(F.Q8b) We prove that \( \gamma'(1) \) is perpendicular to \( T_{p_\infty} W \) by contradiction. Suppose vector \( V_s \in T_{p_\infty} W \) satisfies \( g(V_s, \gamma'(1)) < 0 \). We may choose a smooth vector field \( V(t) \) along \( \gamma(t) \) such that \( V(0) = 0 \) and \( V(1) = V_s \). Let \( \gamma_s, s \in (-\epsilon, \epsilon) \), be a variation of \( \gamma \) with variational field \( V(t) \).

By the first variation formula of length functional around a minimal geodesic we have
\[
0 \leq \frac{d}{ds} \bigg|_{s=0} L(\gamma_s) = g(V(1), \gamma'(1)) - g(V(0), \gamma'(0)) = g(V_s, \gamma'(1)) < 0,
\]
which is the required contradiction.

The perpendicular property at \( t = 0 \) can be proved similarly.

\( \square \)

**F.Q9.** Consider torus \( T^3 = S^1 \times S^1 \times S^1 \). Below we view each \( S^1 \) as \([-\pi, \pi]\) with \(-\pi\) and \(\pi\) identified.

(F.Q9a) Does it admit a metric with positive Ricci curvature everywhere?

(F.Q9b) If not above, construct (not-necessarily in a closed formula) a smooth metric \( g \) on \( T^3 \) such that all the sectional curvature is \(+1\) on the following sub-domain
\[
(-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \subset T^3.
\]

**F.Q9 Sol.** (F.Q9a) No. To see this by contradiction, we assume that the Ricci curvature is positive on \( T^3 \). Since \( T^3 \) is compact, we may assume that Ricci curvature has a constant lower bound \( k > 0 \). By the Myers theorem the universal covering space of \( T^3 \) should be compact, which is a contradiction.

(F.Q9b) We may choose a diffeomorphism \( \psi : (-\pi/3, \pi/3) \times (-\pi/3, \pi/3) \times (-\pi/3, \pi/3) \subset T^3 \) to an open set in sphere \( S^4 \) of radius 1 with standard metric \( g_{sph} \). Let \( \rho \) be a smooth cutoff function which equals to 1 on \((-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \times (-\pi/4, \pi/4)\) and has support inside \((-\pi/3, \pi/3) \times (-\pi/3, \pi/3) \times (-\pi/3, \pi/3)\).

Let \( g_0 \) be an arbitrary Riemannian metric on \( T^3 \), then it is easy to see
\[
\rho \psi^* g_{sph} + (1 - \rho) g_0
\]
is the required metric \( g \).
**F.Q10**. Let $G^n$ be a Lie group of dimension $n$, with identity element $e$. Recall that multiplication from the left by an element $h \in G$ defines a diffeomorphism $L_h : G \to G, \ L_h(g) = h \cdot g$. A differential form $\tilde{\omega}$ on $G$ is said to be left invariant if it satisfies the condition $(L_h)^* \tilde{\omega} = \tilde{\omega}$ for all $h \in G$.

(F.Q10a) Prove that any covector $v^* \in T^*_e G$ uniquely extends to a smooth, left-invariant 1-form $\omega$ on $G$.

(F.Q10b) Use the result of (F.Q10a) to prove that there exist $n$ pointwise independent, left-invariant 1-forms $\omega^k$ on $G, \ k = 1, 2, \ldots, n$.

(F.Q10c) Prove that there exist constants $c^k_{ij}$, such that for each $\omega^k$ in (F.Q10b)

$$d\omega^k = \sum_{1 \leq i<j \leq n} c^k_{ij} : \omega^i \wedge \omega^j.$$  

**F.Q10 Sol.** (F.Q10a) We define $\omega$ by $\omega(g) = (L_{g^{-1}})^* v^* \in T^*_g G$ for each $g \in G$. From the definition of Lie group, operator $(L_{g^{-1}})^*$ is smooth in $g$, hence $\omega$ is a smooth 1-form on $G$.

To see that $\omega$ is left invariant, we compute

\begin{align*}
((L_h)^* \omega)(g) &= (L_h)^* (\omega(hg)) = (L_h)^* (L_{(hg)^{-1}})^* v^* = (L_{(hg)^{-1}} \circ L_h)^* v^* \\
 &= (L_{(hg)^{-1}} \circ L_h)^* v^* = (L_{g^{-1}})^* v^* = \omega(g).
\end{align*}

The uniqueness is obvious.

(F.Q10b) Let $\{e^*_i\}_{i=1}^n$ be a coframe in $T^*_e G$. Then from (F.Q10a) they define left-invariant 1-forms $\{\omega^k\}_{k=1}^n$. Since $(L_{g^{-1}})^* : T^*_e G \to T^*_g G$ is an isomorphism of vector spaces, we conclude that $\omega^i(g) = (L_{g^{-1}})^* e^*_i, \cdots, \omega^n(g) = (L_{g^{-1}})^* e^*_n$ are linearly independent.

(F.Q10c) From (F.Q10b) we know that $\omega^i(g) \wedge \omega^j(g), 1 \leq i < j \leq n$ form a basis of $\Lambda^2 T_g G$ for each $g \in G$. Hence there are functions $c^k_{ij}(g)$ such that

$$d\omega^k(g) = \sum_{1 \leq i<j \leq n} c^k_{ij}(g) : \omega^i(g) \wedge \omega^j(g),$$  

(1)

$$d\omega^k(hg) = \sum_{1 \leq i<j \leq n} c^k_{ij}(hg) : \omega^i(hg) \wedge \omega^j(hg),$$

for all $g, h \in G$. Applying $L_h^*$ to the second equation above we get

$$d(L_h^* \omega^k(hg)) = \sum_{1 \leq i<j \leq n} c^k_{ij}(hg) : L_h^* \omega^i(hg) \wedge L_h^* \omega^j(hg)$$

$$d\omega^k(g) = \sum_{1 \leq i<j \leq n} c^k_{ij}(hg) : \omega^i(g) \wedge \omega^j(g).$$

Hence from (1) we conclude $c^k_{ij}(hg) = c^k_{ij}(g)$, i.e., $c^k_{ij}$ is a constant function. \qedsymbol