Qualifying Exam, Differential Geometry
Fall 2020

F.Q1. Consider the map

$$f : \mathbb{C}^2 \to \mathbb{R}^2, \qquad f(z,w) = (|z|^2 + |w|^2, 2|z|^2 - 3|w|^2).$$

Prove that pre-image $f^{-1}(1,0) \subset \mathbb{C}^2$ is a smooth, compact, 2-dimensional manifold.

F.Q1 Sol. Use real coordinates (x,y,u,v) where $z = x+iy$ and $w = u+iv$, the Jacobian of map

$$f(x,y,u,v) = (x^2+y^2+u^2+v^2, 2x^2+2y^2-3u^2-3v^2$$

is given by

$$Df = \begin{bmatrix} 2x & 2y & 2u & 2v \\ 4x & 4y & -6u & -6v \end{bmatrix}.$$

If Df has rank ≤ 1 on $f^{-1}(1,0)$ we would have

$$-20xu = 0, \quad -20xv = 0, \quad -20yu = 0, \quad -20yv = 0,$$

where the first equality is the determinant of the submatrix formed by the first and third columns. This implies $(x^2+y^2)(u^2+v^2) = 0$, which contradicts with $x^2+y^2+u^2+v^2 = 1$. Hence Df has rank 2 on $f^{-1}(1,0)$, and by the transversality theorem we conclude that $f^{-1}(1,0)$ is a smooth 2-dimensional manifold. The compactness of $f^{-1}(1,0)$ follows from that $f^{-1}(1,0)$ is a subset of $S^3(1)$. □

F.Q2. Let $\mathbb{R}\mathrm{P}^2 \equiv S^2/\sim$ be the real projective space. Let $f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3$ be a map defined by

$$f([x,y,z]) = \frac{1}{x^2+y^2+z^2}(yz, zx, xy).$$

(F.Q2a) Show that f is smooth.

(F.Q2b) Find a point in $\mathbb{R}\mathrm{P}^2$ to show that f is not an immersion.

F.Q2 Sol. (F.Q2a) We may cover S^2 (and hence for $\mathbb{R}\mathrm{P}^2$) by $S_{x,+} = \{(x,y,z),\, x > 0, x^2+y^2+z^2 = 1$ and 5 others similarly defined. Then $\varphi_{x,+} : S_{x,+} \to (y,z)$ and 5 others defines a smooth coordinate atlas on $\mathbb{R}\mathrm{P}^2$.

To see that f is smooth it is sufficient to check

$$f \circ \varphi_{x,+}^{-1}(y,z) = (yz, z\sqrt{1-y^2-z^2}, y\sqrt{1-y^2-z^2})$$

is smooth on $\{(y,z),\, y^2+z^2 < 1\}$, and 5 others is smooth also. This is clearly true for $f \circ \varphi_{x,+}^{-1}$ and for 5 others by symmetry.

(F.Q2b) At point $(y,z) = (0,0)$ we compute the Jacobian of $f \circ \varphi_{x,+}^{-1}$ and find it is a zero matrix. Hence f can not be an immersion. □

F.Q3. Let $F : S^3 \to S^2$ be a smooth map between spheres.

(F.Q3a) Show that there exist a smooth 2-form ω on S^2 such that $\int_{S^2} \omega = 1$, and a smooth 1-form η on S^3 such that $F^*\omega = d\eta$.

(F.Q3b) Let $\tilde{\omega}$ be another smooth 2-form on S^2 satisfying $\int_{S^2} \tilde{\omega} = 1$. Show that there is a smooth 1-form τ on S^2 such that $\tilde{\eta} \doteqdot \eta + F^*\tau$ satisfies $F^*\tilde{\omega} = d\tilde{\eta}$.

(F.Q3c) Show that

$$\int_{S^3} \eta \wedge d\eta = \int_{S^3} \tilde{\eta} \wedge d\tilde{\eta}.$$

F.Q3 Sol. (F.Q3a) Choose local coordinate chart with $S_{x,+} = \{(x, y, z),\, x > 0, x^2 + y^2 + z^2 = 1$ and $\varphi_{x,+} : S_{x,+} \to (y, z)$. Let ρ by a cutoff function supported on $S_{x,+}$. Then $\rho dy \wedge dz$ is a smooth 2-form on S^2 with $\int_{S^2} \rho dy \wedge dz > 0$. We may choose

$$\omega = \frac{1}{\int_{S^2} \rho dy \wedge dz} \cdot \rho dy \wedge dz.$$

Note that $d\omega = 0$.

Note that $dF^*\omega = F^*d\omega = F^*0 = 0$ and cohomology class $[F^*\omega] \in H^2_{\text{DeRh}}(S^3) = 0$, hence there is a smooth 1-form η on S^3 such that $F^*\omega = d\eta$.

(F.Q3b) Since as cohomology class $[\omega] = [\tilde{\omega}] \in H^2_{\text{DeRh}}(S^2) = \mathbb{R}$, there is a smooth 1-form τ on S^2 such that $\tilde{\omega} = \omega + d\tau$. Define $\tilde{\eta} = \eta + F^*\tau$, we verify

$$F^*\tilde{\omega} = F^*(\omega + d\tau) = d\eta + dF^*\tau = d\tilde{\eta}.$$

(F.Q3c) Note that

$$\eta \wedge dF^*\tau = -d(\eta \wedge F^*\tau) + d\eta \wedge F^*\tau$$

we compute

$$\begin{aligned}
&\int_{S^3} \tilde{\eta} \wedge d\tilde{\eta} - \int_{S^3} \eta \wedge d\eta \\
=&\int_{S^3} \eta \wedge dF^*\tau + F^*\tau \wedge d\eta + F^*\tau \wedge dF^*\tau \\
=&\int_{S^3} -d(\eta \wedge F^*\tau) + 2d\eta \wedge F^*\tau + F^*\tau \wedge dF^*\tau \\
=&\int_{S^3} 2F^*\omega \wedge F^*\tau + F^*\tau \wedge dF^*\tau \quad \text{by Stokes theorem} \\
=&\int_{S^3} F^*(2\omega \wedge \tau + \tau \wedge d\tau) \\
=&0,
\end{aligned}$$

where the last equality is due to $0 = 2\omega \wedge \tau + \tau \wedge d\tau \in \Omega^3(S^2)$. □

F.Q4. Let $M^3 \to \mathbb{R}^3$ be a compact, connected, 3-dimensional smooth submanifold/domain with boundary ∂M which has the induced (Stokes) orientation. For any point $p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, define the translated submanifold

$$M_{p_0} \doteqdot \{(x + x_0, y + y_0, z + z_0),\, (x, y, z) \in M\}.$$

Compute the limit

$$\lim_{p_0\to\infty}\int_{\partial M_{p_0}}\omega$$

where ω is the 2-form

$$\omega=\frac{xdy\wedge dz+ydz\wedge dx+zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}.$$

F.Q4 Sol. First we show $d\omega=0$ by computing

$$\begin{aligned}d\omega=&\frac{dx\wedge dy\wedge dz+dy\wedge dz\wedge dx+dz\wedge dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}\\&-\frac{3(xdx+ydy+zdz)}{(x^2+y^2+z^2)^{5/2}}\wedge(xdy\wedge dz+ydz\wedge dx+zdx\wedge dy)\\=&0.\end{aligned}$$

Note that for p_0 far away from origin $\vec{0}$ we have $\vec{0}\notin M_{p_0}$. Since ω is smooth on $\mathbb{R}^3\setminus\{0\}$, we have

$$\int_{\partial M_{p_0}}\omega=\int_{M_{p_0}}d\omega=0$$

by Stokes theorem. The limit follows. □

F.Q5. Consider two Riemannian metrics g and $u^2\cdot g$ on some smooth manifold M^n, where u is a smooth positive function on M. Let (U,φ,x) be a local coordinate chart on M. Compute the Christoffel symbols $\Gamma(u^2\cdot g)_{ij}^k$ of Riemannian connection associated to metric $u^2\cdot g$ in terms of u (along with its derivatives) and the Christoffel symbols $\Gamma_{ij}^k=\Gamma(g)_{ij}^k$.

F.Q5 Sol. We compute

$$\begin{aligned}\Gamma(u^2\cdot g)_{ij}^k=&\frac{1}{2}(u^2\cdot g)^{kl}\left(\partial_i(u^2\cdot g)_{jl}+\partial_j(u^2\cdot g)_{il}-\partial_l(u^2\cdot g)_{ij}\right)\\=&\frac{1}{2}u^{-2}\cdot g^{kl}\left(u^2\partial_ig_{jl}+u^2\partial_jg_{il}-u^2\partial_lg_{ij}+2uu_ig_{jl}+2uu_jg_{il}-2uu_lg_{ij}\right)\\=&\Gamma(g)_{ij}^k+\frac{1}{u}\delta_j^ku_i+\frac{1}{u}\delta_i^ku_j-\frac{1}{u}g^{kl}g_{ij}u_l,\end{aligned}$$

where $u_i=\partial_iu=\frac{\partial u}{\partial x^i}$. □

F.Q6. Consider parametrized surface $\Sigma^2\subset\mathbb{R}^3$ defined by

$$x=3u-u^3+3uv^2,\quad y=-3v+v^3-3u^2v,\quad z=3u^2-3v^2,$$

where $(u,v)\in\mathbb{R}^2$. Show that its first fundamental form I (induced metric) is rotationally symmetric. I.e., there is a change of coordinates $(u,v)\to(w,\theta)$ such that

$$I=h(w)^2dw^2+k(w)^2d\theta^2,$$

where $h(w)$ and $k(w)$ are two positive functions of w.

F.Q6 Sol. We compute

$$\begin{aligned} I =&((3-3u^2+3v^2)du+6uvdv)^2+(-6uvdu+(-3+3v^2-3u^2)dv)^2+(6udu-6vdv)^2 \\ =&9(1+u^2+v^2)^2du^2+9(1+u^2+v^2)^2dv^2 \end{aligned}$$

Let $u = w\cos\theta$ and $v = w\sin\theta$, then

$$I = 9(1+w^2)^2(dw^2+w^2d\theta^2),$$

which is rotationally symmetric. □

F.Q7. Suppose the metric on some surface Σ^2 is given by

$$ds^2 = du^2 + 2\cos f(u,v)dudv + dv^2,$$

where f is a function of (u,v) taking value in $(-1,1)$. Show that the Gauss curvature $K = -\frac{\frac{\partial^2 f}{\partial u \partial v}}{\sin f}$.

F.Q7 Sol. From the following formula for Gauss curvature

$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv}+F_{uv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u-\frac{1}{2}E_v \\ F_v-\frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG-F^2)^2}$$

we have

$$\begin{aligned} K =& \frac{\begin{vmatrix} -\cos f\cdot f_uf_v-\sin f\cdot f_{uv} & 0 & -\sin f\cdot f_u \\ -\sin f\cdot f_v & 1 & \cos f \\ 0 & \cos f & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & \cos f \\ 0 & \cos f & 1 \end{vmatrix}}{(1-\cos^2 f)^2} \\ =& \frac{(-\cos f\cdot f_uf_v-\sin f\cdot f_{uv})\cdot \begin{vmatrix} 1 & \cos f \\ \cos f & 1 \end{vmatrix} + \sin f\cdot f_v\cdot \begin{vmatrix} 0 & -\sin f\cdot f_u \\ \cos f & 1 \end{vmatrix}}{\sin^4 f} \\ =& -\frac{f_{uv}}{\sin f}. \end{aligned}$$

□

F.Q8. Let (M^n, g) be a complete Riemannian manifold. Let N^k and W^l be two embeded submanifolds of M with no boundary, $1 \le k, l \le n-1$. Assume (i) N is compact, (ii) W is a closed subset (not necessarily compact), and (iii) $N \cap W = \emptyset$.

(F.Q8a) Prove that there exists a minimal geodesic $\gamma : [0,1] \to M$ with $\gamma(0) \in N$ and $\gamma(1) \in W$ such that length

$$L(\gamma) = \inf_{p\in N \text{ and } q\in W} d_M(p,q),$$

where d_M is the distance function on M induced from metric g.

(F.Q8b) Prove that the geodesic γ in part (F.Q8a) is perpendicular to N at $t = 0$, and to W at $t = 1$.

F.Q8 Sol. (F.Q8a) Let $\{(p_i, q_i)\}$ be a sequence of point pairs in $N \times W$ such that

$$d_M(p_i, q_i) \to \inf_{p \in N \text{ and } q \in W} d_M(p, q) \quad \text{as } i \to \infty.$$

Since N is compact, there is a subsequence (still indexed by i) such that $p_i \to p_\infty \in N$. Hence the corresponding sequence $\{q_i\}$ is in a bounded neighborhood of p_∞. Since M is complete, by taking another subsequence we may assume $q_i \to q_\infty$. Since W is closed we have $q_\infty \in W$, hence

$$0 < d_M(p_\infty, q_\infty) = \inf_{p \in N \text{ and } q \in W} d_M(p, q)$$

We may take $\gamma : [0, 1] \to M$ to be a minimal geodesic joining p_∞ and q_∞.

(F.Q8b) We prove that $\gamma'(1)$ is perpendicular to $T_{q_\infty} W$ by contradiction. Suppose vector $V_* \in T_{q_\infty} W$ satisfies $g(V_*, \gamma'(1)) < 0$. We may choose a smooth vector field $V(t)$ along $\gamma(t)$ such that $V(0) = 0$ and $V(1) = V_*$. Let γ_s, $s \in (-\epsilon, \epsilon)$, be a variation of γ with variational field $V(t)$.

By the first variation formula of length functional around a minimal geodesic we have

$$0 \leq \frac{d}{ds}\bigg|_{s=0} L(\gamma_s) = g(V(1), \gamma'(1)) - g(V(0), \gamma'(0)) = g(V_*, \gamma'(1)) < 0,$$

which is the required contradiction.

The perpendicular property at $t = 0$ can be proved similarly. □

F.Q9. Consider torus $T^3 = S^1 \times S^1 \times S^1$. Below we view each S^1 as $[-\pi, \pi]$ with $-\pi$ and π identified.

(F.Q9a) Does it admit a metric with positive Ricci curvature everywhere?

(F.Q9b) If not above, construct (not-necessarily in a closed formula) a smooth metric g on T^3 such that all the sectional curvature is $+1$ on the following sub-domain

$$(-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \subset T^3.$$

F.Q9 Sol. (F.Q9a) No. To see this by contradiction, we assume that the Ricci curvature is positive on T^3. Since T^3 is compact, we may assume that Ricci curvature has a constant lower bound $k > 0$. By the Myers theorem the universal covering space of T^3 should be compact, which is a contradiction.

(F.Q9b) We may choose a diffeomorphism $\psi : (-\pi/3, \pi/3) \times (-\pi/3, \pi/3) \times (-\pi/3, \pi/3) \subset T^3$ to an open set in sphere S^3 of radius 1 with standard metric g_{sph}. Let ρ be a smooth cutoff function which equals to 1 on $(-\pi/4, \pi/4) \times (-\pi/4, \pi/4) \times (-\pi/4, \pi/4)$ and has support inside $(-\pi/3, \pi/3) \times (-\pi/3, \pi/3) \times (-\pi/3, \pi/3)$.

Let g_0 be an arbitrary Riemannian metric on T^3, then it is easy to see

$$\rho \psi^* g_{sph} + (1 - \rho) g_0$$

is the required metric g. □

F.Q10. Let G^n be a Lie group of dimension n, with identity element e. Recall that multiplication from the left by an element $h \in G$ defines a diffeomorphism $L_h : G \to G$, $L_h(g) = h \cdot g$. A differential form $\tilde{\omega}$ on G is said to be **left invariant** if it satisfies the condition $(L_h)^*\tilde{\omega} = \tilde{\omega}$ for all $h \in G$.

(F.Q10a) Prove that any covector $v^* \in T_e^*G$ uniquely extends to a smooth, left-invariant 1-form ω on G.

(F.Q10b) Use the result of (F.Q10a) to prove that there exist n pointwise independent, left-invariant 1-forms ω^k on G, $k = 1, 2, \cdots, n$.

(F.Q10c) Prove that there exist constants c_{ij}^k, such that for each ω^k in (F.Q10b)

$$d\omega^k = \sum_{1 \le i < j \le n} c_{ij}^k \cdot \omega^i \wedge \omega^j.$$

F.Q10 Sol. (F.Q10a) We define ω by $\omega(g) = (L_{g^{-1}})^* v^* \in T_g^*G$ for each $g \in G$. From the definition of Lie group, operator $(L_{g^{-1}})^*$ is smooth in g, hence ω is a smooth 1-form on G.

To see that ω is left invariant, we compute

$$\begin{aligned}((L_h)^*\omega))(g) =& (L_h)^*(\omega(hg)) = (L_h)^* \circ (L_{(hg)^{-1}})^* v^* = (L_{(hg)^{-1}} \circ L_h)^* v^* \\ =& (L_{(hg)^{-1} \cdot h})^* v^* = (L_{g^{-1}})^* v^* = \omega(g).\end{aligned}$$

The uniqueness is obvious.

(F.Q10b) Let $\{e_i^*\}_{k=1}^n$ be a coframe in T_e^*G. Then from (F.Q10a) they define left-invariant 1-forms $\{\omega^k\}_{k=1}^n$. Since $(L_{g^{-1}})^* : T_e^*G \to T_g^*G$ is an isomorphism of vector spaces, we conclude that $\omega^1(g) = (L_{g^{-1}})^* e_1^*, \cdots, \omega^n(g) = (L_{g^{-1}})^* e_n^*$ are linearly independent.

(F.Q10c) From (F.Q10b) we know that $\omega^i(g) \wedge \omega^j(g)$, $1 \le i < j \le n$ form a basis of $\Lambda^2 T_g G$ for each $g \in G$. Hence there are functions $c_{ij}^k(g)$ such that

$$d\omega^k(g) = \sum_{1 \le i < j \le n} c_{ij}^k(g) \cdot \omega^i(g) \wedge \omega^j(g), \tag{1}$$

$$d\omega^k(hg) = \sum_{1 \le i < j \le n} c_{ij}^k(hg) \cdot \omega^i(hg) \wedge \omega^j(hg),$$

for all $g, h \in G$. Applying L_h^* to the second equation above we get

$$\begin{aligned}d(L_h^*\omega^k(hg)) &= \sum_{1 \le i < j \le n} c_{ij}^k(hg) \cdot L_h^*\omega^i(hg) \wedge L_h^*\omega^j(hg) \\ d\omega^k(g) &= \sum_{1 \le i < j \le n} c_{ij}^k(hg) \cdot \omega^i(g) \wedge \omega^j(g).\end{aligned}$$

Hence from (1) we conclude $c_{ij}^k(hg) = c_{ij}^k(g)$, i.e., c_{ij}^k is a constant function. □