ANALYSIS QUALIFYING EXAM FOR FALL 2020

Instructions: Partial credit will be given when appropriate. The decision on this examination will be based not only on the total point score, but also on whether answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot be provided. A completely correct solution is worth more than the same number of points earned in small amounts of partial credit on several problems.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

Problem 1. Let $(X, \mathcal{M})$ be a measurable space and let $f \colon X \to \mathbb{R}$ be a function. For $r \in \mathbb{R}$ define

$$E_r = \{x \in X \colon f(x) < r\}.$$

Suppose $E_r \in \mathcal{M}$ for all $r \in \mathbb{Q}$. Show that f is measurable.

Topic: Measure theory.

Difficulty rating: 1 (scale: 1–6).

Solution. It suffices to show that $E_r \in \mathcal{M}$ for all $r \in \mathbb{R}$. Fix $r \in \mathbb{R}$, and let $(r_n)_{n \in \mathbb{Z}}$ be a nondecreasing sequence in $\mathbb{Q} \cap (-\infty, r)$ such that $\lim_{n \to \infty} r_n = r$. Then

$$E_r = \bigcup_{n=1}^{\infty} E_{r_n}.$$

Since $E_{r_n} \in \mathcal{M}$ for all $n \in \mathbb{Z}_{>0}$ and since $\mathcal{M}$ is a σ-algebra, $E_r \in \mathcal{M}$. $\square$

Problem 2. Let m be Lebesgue measure on $\mathbb{R}$. Let $(f_n)_{n \in \mathbb{Z}_{\geq 0}}$ be a sequence of nonnegative functions in $L^2([0,1], m)$, Suppose that $f_n \to 0$ almost everywhere on $[0,1]$. Show that

$$\lim_{n \to \infty} \int_{(0,1]} \left(\frac{f_n(x)}{[1 + f_n(x)] \sin(x)} \right)^{1/2} dm(x) = 0.$$

Topic: Measure theory.

Difficulty rating: 2 (scale: 1–6).

Solution. For $n \in \mathbb{Z}_{>0}$ and $x \in (0,1]$, set

$$F_n(x) = \left(\frac{f_n(x)}{[1 + f_n(x)] \sin(x)} \right)^{1/2}.$$

Then F_n is measurable for every $n \in \mathbb{Z}_{>0}$, we have $F_n(x) \to 0$ for almost all $x \in (0,1]$, and

$$|F_n(t)| \leq \frac{1}{\sqrt{\sin(x)}}$$

for all $n \in \mathbb{Z}_{>0}$ and $x \in (0,1]$.

We next claim that there is a constant $c > 0$ such that $\sin(x) \geq cx$ for all $x \in [0, 1]$. To see this, observe that $x \mapsto \cos(x)$ is nonincreasing on $[0, \pi]$ and that $\cos(1) > 0$ (since $1 < \frac{\pi}{2}$). Therefore, for all $x \in [0, 1]$, we have $\cos(x) \geq \cos(1)$, so

$$\sin(x) = \int_0^x \cos(t)\,dt \geq \cos(1)x.$$

The claim is proved, with $c = \cos(1)$.

It follows that for all $n \in \mathbb{Z}_{>0}$ and $x \in (0, 1]$,

$$|F_n(t)| \leq \frac{1}{\sqrt{cx}}.$$

Since $x \mapsto 1/\sqrt{cx}$ is integrable on $(0, 1]$, we can apply the Dominated Convergence Theorem to get

$$\lim_{n\to\infty} \int_{(0,1]} F_n\,dm = \int_{(0,1]} \Big(\lim_{n\to\infty} F_n(x) \Big)\,dm(x) = \int_{(0,1]} 0\,dm = 0.$$

This completes the solution. □

Alternate solution. This solution differs only in the method used to find the constant c in the second paragraph.

The function $x \mapsto \sin(x)$ is concave ("concave down" in the terminology of freshman calculus books) on $[0, \pi]$, so on $[0, 1]$. Therefore for all $x \in [0, 1]$,

$$\sin(x) \geq (1 - x)\sin(0) + x\sin(1) = \sin(1)x.$$

Since $1 < \frac{\pi}{2}$, we have $\sin(1) > 0$, so $c = \sin(1)$ works. □

Second alternate solution. This solution differs only in the method used to find the constant c in the second paragraph.

Since $\lim_{x\to 0} \sin(x)/x = 1$, there is $\delta > 0$ such that for $x \in (0, \delta)$ we have $\sin(x)/x > \frac{1}{2}$. Without loss of generality $\delta \leq 1$. Since $x \mapsto \sin(x)$ is nondecreasing on $\left[0, \frac{\pi}{2}\right]$, for $x \in [\delta, 1]$ we have

$$\frac{\sin(x)}{x} \geq \sin(x) \geq \sin(\delta).$$

Therefore $c = \min\big(\sin(\delta), \frac{1}{2}\big)$ works. □

As is clear, there are many ways to produce a bounding function for use in the Dominated Convergence Theorem. This step is a significant part of the solution, and an actual argument must be given. For example, in the second alternate solution, it is not enough to say that $\lim_{x\to 0} \sin(x)/x = 1$ and leave it at that. (One reason, but not the only reason, that this is insufficient is that it doesn't say anything about possible vanishing of the denomination on $(0, 1]$.)

Some freshman calculus books have a version of the Limit Comparison Test for improper integrals, which can also be used here.

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of functions in $L^2(X, \mu)$ and and let f also be a function in $L^2(X, \mu)$. Suppose that

$$\lim_{n\to\infty} \int_X |f_n - f|^2\,d\mu = 0.$$

Prove that for any $\varepsilon > 0$, there is $\delta > 0$ satisfying the following. For any measurable set $E \subset X$ with $\mu(E) < \delta$, and any $n \in \mathbb{Z}_{>0}$,

$$\int_E |f_n|^2 \, d\mu < \varepsilon.$$

Topic: Measure theory and functional analysis.

Difficulty rating: 1 (scale: 1–6).

Solution. Define measures μ_n for $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ by, for all $E \in \mathcal{M}$, $\mu_n(E) = \int_E |f_n|^2 \, d\mu$ for $n \in \mathbb{Z}_{>0}$ and $\mu_\infty(E) = \int_E |f|^2 \, d\mu$. By (a version of) the Radon-Nikodym Theorem, for $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, μ_n is a positive measure which is absolutely continuous with respective to μ. Therefore, for $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, there is $\delta_n > 0$ such that, for any $E \in \mathcal{M}$ with $\mu(E) < \delta_n$, we have $\mu_n(E) < \frac{\varepsilon}{4}$.

By hypothesis there is an integer $N \geq 1$ such that, for all $n \geq N$,

$$\|f_n - f\|_2^2 < \frac{\varepsilon}{4}.$$

In particular, for any $E \in \mathcal{M}$,

$$\left(\int_E |f_n - f|^2 \, d\mu \right)^{1/2} \leq \|f_n - f\|_2 < \left(\frac{\varepsilon}{4} \right)^{1/2}.$$

If $n \geq N$ and $\mu(E) < \delta_\infty$, then

$$\left(\int_E |f_n|^2 \, d\mu \right)^{1/2} \leq \left(\int_E |f_n - f|^2 \, d\mu \right)^{1/2} + \left(\int_E |f|^2 \, d\mu \right)^{1/2} < 2 \left(\frac{\varepsilon}{4} \right)^{1/2}.$$

Choose $\delta = \min(\delta_\infty, \delta_1, \delta_2, ..., \delta_N) > 0$. Then, whenever $\mu(E) < \delta$, for all n,

$$\int_E |f_n|^2 \, d\mu < \varepsilon.$$

This completes the solution. □

The fact that $E \mapsto \int_E g \, d\mu$ defines a measure is Theorem 1.39 in Rudin's book. The fact that if $\mu \ll \nu$ then for every $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < \varepsilon$ is Theorem 6.11(b) in Rudin's book. It is also a standard exercise (but not officially stated as a theorem, except as a consequence of Theorem 6.11) that if $g \in L^1(\mu)$ then for every $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |g| \, d\mu < \varepsilon$.

Problem 4. Show that there is a regular complex Borel measure μ on $[0, 1]$ such that

$$\int_{[0,1]} t^k \, d\mu(t) = \frac{k}{2^{k-1}}$$

for $k = 1, 2, \ldots, 2020$.

Topic: Measure theory and functional analysis.

Difficulty rating: 2 (scale: 1–6).

Solution. Let $E \subset C([0,1])$ be the subspace spanned by the functions $f_k(t) = t^k$ for $k = 1, 2, \ldots, 2020$.

We claim that $f_1, f_2, \ldots, f_{2020}$ are linearly independent. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_{2020} \in \mathbb{C}$ and $\sum_{k=1}^{2020} \alpha_k f_k$ is the zero element of $C([0,1])$. The function $p(t) = \sum_{k=1}^{2020} \alpha_k f_k(t)$ is then a polynomial with infinitely many roots, so it is the zero polynomial, and $\alpha_1 = \alpha_2 = \cdots = \alpha_{2020} = 0$. This proves the claim.

Given the claim, there is a linear functional $\omega_0 \colon E \to \mathbb{C}$ such that

$$\omega_0(f_k) = \frac{k}{2^{k-1}}$$

for $k = 1, 2, \ldots, 2020$. Since E is finite dimensional, ω_0 is bounded. By the Hahn-Banach theorem, there is a bounded linear functional $\omega \colon C([0,1]) \to \mathbb{C}$ such that $\omega|_E = \omega_0$. It follows from the Riesz representation theorem for spaces of the form $C(X)$ that there is a regular complex Borel measure μ on $[0,1]$ such that $\omega(f) = \int_{[0,1]} f \, d\mu$ for all $f \in C([0,1])$. In particular,

$$\int_{[0,1]} t^k \, d\mu(t) = \omega_0(f_k) = \frac{k}{2^{k-1}}$$

for $k = 1, 2, \ldots, 2020$. $\square$

To be complete, a solution must give a reason that the functions $t, t^2, t^3, \ldots, t^{2020}$ are linearly independent, or give some other reason that the linear functional ω_0 exists.

It is also possible to directly prove the existence of a suitable measure μ which is a linear combination of 2020 point masses.

Problem 5. Let m be Lebesgue measure on $\mathbb{R}$. Let F be a nondecreasing function on $[a,b]$. Show that its derivative F' is in $L^1([0,1])$ and

$$\int_a^b F' dm \leq F(b) - F(a).$$

Further show that if there is $t \in (a,b)$ such that F is not continuous at t, then the inequality above is strict.

Topic: Measure theory.

Difficulty rating: 2 (scale: 1–6).

This solution assumes it is known that monotone functions are differentiable almost everywhere, something which is sometimes, but not always, in the course.

Solution. Extend F to $[a-1,\, b+1]$ by defining $F(x) = F(a)$ if $x \in [a-1,\, a]$ and $F(x) = F(b)$ if $x \in [b,\, b+1]$. For each integer $n \geq 1$, define

$$\varphi_n(x) = \frac{F(x + 1/n) - F(x)}{1/n}$$

for $x \in [a,b]$.

The function F is measurable because it is nondecreasing, and therefore integrable on $[a-1,\, b+1]$, since

$$\int_{a-1}^{b+1} |F| \, dm \leq (b-a-2) \max\bigl(|F(a-1)|,\, |F(b+1)|\bigr) = (b-a-2) \max\bigl(|F(a)|,\, |F(b)|\bigr).$$

So $\varphi_n \in L^1([a,b])$ for all $n \in \mathbb{Z}_{>0}$. Moreover,

$$\begin{aligned}\int_a^b \varphi_n(x)\,dx &= n\left(\int_a^b F(x+1/n)\,dx - \int_a^b F(x)\,dx\right)\\ &= n\left(\int_b^{b+1/n} F(x)\,dx - \int_a^{a+1/n} F(x)\,dx\right)\\ &\leq n\left(F(b)\int_b^{b+1} dx - F(a)\int_a^{a+1/n} dx\right) = F(b)-F(a).\end{aligned}$$

The functions φ_n are nonnegative (because F is nondecreasing) and converge pointwise almost everywhere to F' (since F' exists almost everywhere). Moreover, by the definition of the derivative, $F'(x) \geq 0$ almost everywhere. Fatou's Lemma therefore implies that

$$\int_a^b F'(x)\,dx \leq \liminf_{n\to\infty} \int_a^b \varphi_n(x)\,dx \leq F(b)-F(a).$$

This is the first part of the statement.

For the last part of the statement, define $G\colon [a,t] \to \mathbb{R}$ and $H\colon [t,b] \to \mathbb{R}$ by

$$G(x) = \begin{cases} F(x) & x \in [a,t) \\ \lim_{s\to t^-} F(s) & x = t \end{cases} \quad \text{and} \quad H(x) = \begin{cases} \lim_{s\to t^+} F(s) & x = t \\ F(x) & x \in (t,b]. \end{cases}$$

The functions G and H are nondecreasing. Using the first part on these functions at the third step, and the fact that F is nondecreasing but discontinuous at t at the last step,

$$\begin{aligned}\int_a^b F'(x)\,dx &= \int_a^t F'(x)\,dx + \int_t^b F'(x)\,dx = \int_a^t G'(x)\,dx + \int_t^b F'(x)\,dx\\ &\leq \lim_{s\to t^-} F(s) - F(a) + F(b) - \lim_{s\to t^+} F(s) < F(b)-F(a).\end{aligned}$$

This completes the solution. □

Problem 6. Let $p \in [1,\infty)$. The space l^p is defined by

$$l^p = \left\{ (\xi(k))_{k\in\mathbb{Z}_{>0}} \colon \xi(k) \in \mathbb{C} \text{ for all } k \in \mathbb{Z}_{>0} \text{ and } \textstyle\sum_{k=1}^{\infty} |\xi(k)|^p < \infty \right\}.$$

This space is a complex Banach space with the norm

$$\|(\xi(k))_{k\in\mathbb{Z}_{>0}}\|_p = \left(\sum_{k=1}^{\infty} |\xi(k)|^p\right)^{1/p}.$$

(You may use this fact without proof.) Let

$$S = \left\{ (\xi(k))_{k\in\mathbb{Z}_{>0}} \in l^p \colon |\xi(k)| \leq 1/2^k \text{ for all } k \in \mathbb{Z}_{>0} \right\}.$$

Show that S is compact.

Topic: Functional analysis.

Difficulty rating: 3 (scale: 1–6).

Solution. We prove that S is complete and totally bounded. For any metric space X, any $x \in X$, and any $r > 0$, we let

$$B_{r,X}(x) = \big\{ y \in X \colon d(y,x) < r \big\}$$

denote the open ball of radius r about x. Recall that X is totally bounded if for all $\varepsilon > 0$ there is a finite set $S \subset X$ such that $X = \bigcup_{x \in F} B_{\varepsilon,X}(x)$.

For completeness, since l^p is complete, it suffices to show that S is closed in l^p. So let $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in S, let $\xi \in l^p$, and suppose that $\|\xi_n - \xi\|_p \to 0$. For every $k \in \mathbb{Z}_{>0}$, since $|\xi_n(k) - \xi(k)| \leq \|\xi_n - \xi\|_p$, it follows that $\xi_n(k) \to \xi(k)$, so $\xi \in S$. Completeness of S is proved.

To prove that S is totally bounded, let $\varepsilon > 0$. Choose $M \in \mathbb{Z}_{>0}$ so large that

$$\frac{1}{2^M} < \left(1 - \frac{1}{2^p} \right)^{1/p} \varepsilon.$$

This inequality implies

$$\frac{1}{2^{p(M+1)}} \left(\frac{1}{1 - \frac{1}{2^p}} \right) < \frac{\varepsilon^p}{2^p}. \tag{1}$$

Set

$$T = \big\{ \xi \in S \colon \xi(k) = 0 \text{ for } k = M+1,\, M+2,\, \ldots \big\}.$$

Let $Z = \big\{ z \in \mathbb{C} \colon |z| \leq 1 \big\}$. Define $h \colon Z^M \to S$ by

$$h((z_l)_{l \in \mathbb{Z}_{>0}})(k) = \begin{cases} \frac{z_k}{2^k} & k \in \{1, 2, \ldots, M\} \\ 0 & k \in \mathbb{Z}_{>0} \setminus \{1, 2, \ldots, M\}. \end{cases}$$

Then h is a continuous surjective map from the compact space Z^M to the Hausdorff space T, so T is compact. Therefore there is a finite set $F \subset T$ such that

$$T = \bigcup_{\xi \in F} B_{\varepsilon/2,\, T}(\xi).$$

We complete the proof by showing that

$$S = \bigcup_{\xi \in F} B_{\varepsilon,S}(\xi).$$

Let $\eta \in S$. Define $\lambda \in T$ by

$$\lambda(k) = \begin{cases} \eta(k) & k \in \{1, 2, \ldots, M\} \\ 0 & k \in \mathbb{Z}_{>0} \setminus \{1, 2, \ldots, M\}. \end{cases}$$

Choose $\xi \in T$ such that $\lambda \in B_{\varepsilon/2,\, T}(\xi)$. We have, using (1) at the last step,

$$\|\eta - \lambda\|_p^p = \sum_{k=M+1}^{\infty} |\eta(k)|^p \leq \sum_{k=M+1}^{\infty} \frac{1}{2^{kp}} = \frac{1}{2^{p(M+1)}} \left(\frac{1}{1 - \frac{1}{2^p}} \right) < \frac{\varepsilon^p}{2^p}.$$

Therefore

$$\|\eta - \xi\|_p \leq \|\eta - \lambda\|_p + \|\lambda - \xi\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $\eta \in B_{\varepsilon,S}(\xi)$. $\square$

Alternate solution. We first claim that if a sequence $(\xi_n)_{n\in\mathbb{Z}_{>0}}$ in S converges pointwise to $\xi \in S$, then $\|\xi_n - \xi\|_p \to 0$. To prove the claim, let $\varepsilon > 0$. Choose $M \in \mathbb{Z}_{>0}$ so large that

$$\frac{1}{2^M} < \frac{1}{2^{1/p}}\left(1 - \frac{1}{2^p}\right)^{1/p}\varepsilon.$$

This inequality implies

$$\frac{1}{2^{pM}}\left(\frac{1}{1-\frac{1}{2^p}}\right) < \frac{\varepsilon^p}{2}. \tag{2}$$

Choose $N \in \mathbb{Z}_{>0}$ so large that for $k = 1, 2, \ldots, M$ and all $n > N$ we have

$$|\xi_n(k) - \xi(k)| < \frac{\varepsilon}{2^{1/p}M^{1/p}}.$$

For all $n \in \mathbb{Z}_{>0}$ and all $k \in \mathbb{Z}_{>0}$ we have

$$|\xi_n(k) - \xi(k)| \leq |\xi_n(k)| + |\xi(k)| \leq \frac{1}{2^{k-1}}. \tag{3}$$

Therefore, for $n > N$, using (3) on the second term at the second step, and using (2) at the second to last step,

$$\begin{aligned}\|\xi_n - \xi\|_p^p &= \sum_{k=1}^{\infty}|\xi_n(k) - \xi(k)|^p < \sum_{k=1}^{M}\frac{\varepsilon^p}{2M} + \sum_{k=M+1}^{\infty}\left(\frac{1}{2^{k-1}}\right)^p \\ &= \frac{\varepsilon^p}{2} + \frac{1}{2^{pM}}\left(\frac{1}{1-\frac{1}{2^p}}\right) < \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p.\end{aligned}$$

So $\|\xi_n - \xi\|_p < \varepsilon$. The claim is proved.

Now let $Z = \{z \in \mathbb{C} : |z| \leq 1\}$. Set $X = Z^{\mathbb{Z}_{>0}}$, with the product topology, which is compact by Tychonoff's Theorem. Define $h : X \to S$ by

$$h((z_l)_{l\in\mathbb{Z}_{>0}})(k) = \frac{z_k}{2^k}.$$

Then h is bijective. The definition of the product topology and the claim above imply that h is continuous. Since X is compact and S is Hausdorff, it follows that S is compact. □

Second alternate solution (outline). Let $(\xi_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence in S; we prove it has a convergent subsequence. Use a Cantor diagonalization argument to find a subsequence, say $(\eta_n)_{n\in\mathbb{Z}_{>0}}$, such that $(\eta_n(k))_{n\in\mathbb{Z}_{>0}}$ converges for all $k \in \mathbb{Z}_{>0}$. Prove, as in the second solution, that $(\eta_n)_{n\in\mathbb{Z}_{>0}}$ then converges in l^p, and prove that S is closed as in the first solution, so that $\lim_{n\to\infty}\eta_n \in S$. □

Problem 7. Let f be a holomorphic function on the set $B = \{z \in \mathbb{C} : |z| < 2\}$ such that $|f(z)| \leq |z|^2$ for all $z \in B$ and $f(1) = 1$. Prove that $f(z) = z^2$ for all $z \in B$.

Topic: Complex analysis.

Difficulty rating: 2 (scale: 1–6).

Solution. Define $h_0 : B \setminus \{0\} \to \mathbb{C}$ by $h_0(z) = z^{-2}f(z)$. The hypotheses imply that $|h_0(z)| \leq 1$ for all $z \in B \setminus \{0\}$. Since h_0 is holomorphic, h_0 has a removable singularity at 0, that is, there exists a holomorphic function $h : B \to \mathbb{C}$ such that $h(z) = h_0(z)$ for all $z \in B \setminus \{0\}$.

By continuity, we have

$$z^2 h(z) = f(z) \qquad \text{and} \qquad |h(z)| \leq 1$$

for all $z \in B$. Since B is connected and $h(1) = 1$, the Maximum Modulus Theorem implies that h is constant. Thus $h(z) = 1$ for all $z \in B$. Therefore $f(z) = z^2 h(z) = z^2$ for all $z \in B$. □

Alternate solution. We have $f(0) = 0$, but f is not the zero function, so f has an isolated zero at 0. Therefore there exist $m \in \mathbb{Z}_{>0}$ and a holomorphic function g on B such that $g(0) \neq 0$ and $f(z) = z^m g(z)$ for all $z \in B$.

We claim that $m \geq 2$. If not, set $r = \frac{1}{2}|g(0)|$. Then $r > 0$ and, by continuity of g, there is $\varepsilon > 0$ such that $|z| < \varepsilon$ implies $|g(z)| > r$. Taking $z_0 = \frac{1}{2}\min(\varepsilon, r)$, we get

$$r|z_0| < |z_0 g(z_0)| = |f(z_0)| \leq |z_0|^2,$$

so that $r < |z_0|$, a contradiction. The claim is proved.

Given the claim, the function $h(z) = z^{m-2}g(z)$ is a holomorphic function on B, and $f(z) = z^2 h(z)$ for all $z \in B$. The hypothesis $|f(z)| \leq |z|^2$ for all $z \in B$ implies that $|h(z)| \leq 1$ for all $z \in B \setminus \{0\}$. Since $B \setminus \{0\}$ is connected and $h(1) = 1$, the Maximum Modulus Theorem implies that h is constant. Thus $h(z) = 1$ for all $z \in B \setminus \{0\}$, and hence, by continuity, for all $z \in B$. Therefore $f(z) = z^2$ for all $z \in B$. □

Remark. In both solutions, one can use the Open Mapping Theorem instead of the Maximum Modulus Theorem. □

Problem 8. Set $B = \{z \in \mathbb{C} \colon |z| < 2\}$. Prove that there is no holomorphic function f on B such that $\left|f(z) - \frac{1}{z}\right| < 1$ for all $z \in B$ with $|z| = 1$.

Topic: Complex analysis.

Difficulty rating: 2 (scale: 1–6).

Solution. The proof is by contradiction. Let f be such a function. Define $g(z) = zf(z)$ for $z \in B$. Then $|g(z) - 1| < 1$ for all $z \in B$ with $|z| = 1$. Rouché's Theorem therefore implies that g and the constant function with value 1 have the same number of zeros in $\{z \in \mathbb{C} \colon |z| < 1\}$, counting multiplicity. Since $g(0) = 0$ and the constant function with value 1 has no zeros, this is a contradiction. □

Alternate solution. The proof is by contradiction. Let f be such a function. Let $\gamma \colon [0, 2\pi] \to \mathbb{C}$ be the closed curve $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. Then

$$\int_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} \left(\frac{1}{e^{it}}\right) ie^{it}\, dt = 2\pi i.$$

Also, at the second to last step using continuity and compactness to get

$$\sup_{|z|=1} \left|f(z) - \frac{1}{z}\right| < 1,$$

we have

$$\left|\int_\gamma f(z)\,dz - \int_\gamma \frac{1}{z}\,dz\right| = \left|\int_0^{2\pi} f(e^{it}) ie^{it}\,dt - \int_0^{2\pi}\left(\frac{1}{e^{it}}\right) ie^{it}\,dt\right|$$
$$\leq \int_0^{2\pi}\left|f(e^{it}) - \frac{1}{e^{it}}\right| |ie^{it}|\,dt < \int_0^{2\pi} 1\,dt = 2\pi.$$

So $\int_\gamma f(z)\,dz \neq 0$. This contradicts Cauchy's Theorem for a convex set. □

Problem 9. Let f be an entire function. Suppose that the set $\{f(z)\colon |z| \geq 1\}$ is not dense in $\mathbb{C}$. Prove that f is a polynomial.

Topic: Complex analysis.

Difficulty rating: 3 (scale: 1–6).

Alternative hypothesis: $f(\mathbb{C})$ is not dense in $\mathbb{C}$.

Solution. Set $B = \{z \in \mathbb{C}\colon |z| < 1\}$. For $z \in \mathbb{C} \setminus \{0\}$, define $g(z) = f\left(\frac{1}{z}\right)$. Then $g(B \setminus \{0\}) \subset f(\mathbb{C} \setminus B)$, so is not dense. Therefore g has either a pole or a removable singularity at 0 (not an essential singularity). Let n be the order of the pole (taking $n = 0$ in the case of a removable singularity). Then there are $c_1, c_2, \ldots, c_n \in \mathbb{C}$ and an entire function h such that

$$g(z) = h(z) + \sum_{k=1}^{n} \frac{c_k}{z^k}$$

for $z \in \mathbb{C} \setminus \{0\}$.

We have

$$\lim_{z\to\infty} g(z) = f(0) \qquad \text{and} \qquad \lim_{z\to\infty} \sum_{k=1}^{n} \frac{c_k}{z^k} = 0.$$

Therefore $\lim_{z\to\infty} h(z) = f(0)$. It follows that h is bounded. By Liouville's Theorem, there is $c_0 \in \mathbb{C}$ such that $h(z) = c_0$ for all $z \in \mathbb{C}$. Then

$$g(z) = \sum_{k=0}^{n} c_k z^{-k}$$

for all $z \in \mathbb{C} \setminus \{0\}$. Therefore

$$f(z) = \sum_{k=0}^{n} c_k z^k$$

for all $z \in \mathbb{C} \setminus \{0\}$. Since f is continuous at 0, this formula also holds for $z = 0$. Thus f is a polynomial. □