ANALYSIS QUALIFYING EXAM FOR FALL 2020

Instructions: Partial credit will be given when appropriate. The decision on this examination will be based not only on the total point score, but also on whether answers turned in are the result of careful thought and show understanding of the situation, even when the full explanation cannot be provided. A completely correct solution is worth more than the same number of points earned in small amounts of partial credit on several problems.

Answer questions as carefully and completely as possible. Do not make formal arguments without mathematical justification. If you use a major theorem, mention it by name and check its hypotheses.

Problem 1. Let $(X, \mathcal{M})$ be a measurable space and let $f\colon X \to \mathbb{R}$ be a function. For $r \in \mathbb{R}$ define

$$E_r = \{x \in X \colon f(x) < r\}.$$

Suppose $E_r \in \mathcal{M}$ for all $r \in \mathbb{Q}$. Show that f is measurable.

Problem 2. Let m be Lebesgue measure on $\mathbb{R}$. Let $(f_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of nonnegative functions in $L^2([0,1],\, m)$, Suppose that $f_n \to 0$ almost everywhere on $[0,1]$. Show that

$$\lim_{n\to\infty} \int_{(0,1]} \left(\frac{f_n(x)}{[1 + f_n(x)]\sin(x)} \right)^{1/2} dm(x) = 0.$$

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of functions in $L^2(X, \mu)$ and and let f also be a function in $L^2(X, \mu)$. Suppose that

$$\lim_{n\to\infty} \int_X |f_n - f|^2 \, d\mu = 0.$$

Prove that for any $\varepsilon > 0$, there is $\delta > 0$ satisfying the following. For any measurable set $E \subset X$ with $\mu(E) < \delta$, and any $n \in \mathbb{Z}_{>0}$,

$$\int_E |f_n|^2 \, d\mu < \varepsilon.$$

Problem 4. Show that there is a regular complex Borel measure μ on $[0,1]$ such that

$$\int_{[0,1]} t^k \, d\mu(t) = \frac{k}{2^{k-1}}$$

for $k = 1, 2, \ldots, 2020$.

Problem 5. Let m be Lebesgue measure on $\mathbb{R}$. Let F be a nondecreasing function on $[a, b]$. Show that its derivative F' is in $L^1([0,1])$ and

$$\int_a^b F' dm \leq F(b) - F(a).$$

Further show that if there is $t \in (a, b)$ such that F is not continuous at t, then the inequality above is strict.

Problem 6. Let $p \in [1, \infty)$. The space l^p is defined by

$$l^p = \left\{ (\xi(k))_{k \in \mathbb{Z}_{>0}} : \xi(k) \in \mathbb{C} \text{ for all } k \in \mathbb{Z}_{>0} \text{ and } \sum_{k=1}^{\infty} |\xi(k)|^p < \infty \right\}.$$

This space is a complex Banach space with the norm

$$\|(\xi(k))_{k \in \mathbb{Z}_{>0}}\|_p = \left(\sum_{k=1}^{\infty} |\xi(k)|^p \right)^{1/p}.$$

(You may use this fact without proof.) Let

$$S = \left\{ (\xi(k))_{k \in \mathbb{Z}_{>0}} \in l^p : |\xi(k)| \leq 1/2^k \text{ for all } k \in \mathbb{Z}_{>0} \right\}.$$

Show that S is compact.

Problem 7. Let f be a holomorphic function on the set $B = \{ z \in \mathbb{C} : |z| < 2 \}$ such that $|f(z)| \leq |z|^2$ for all $z \in B$ and $f(1) = 1$. Prove that $f(z) = z^2$ for all $z \in B$.

Problem 8. Set $B = \{ z \in \mathbb{C} : |z| < 2 \}$. Prove that there is no holomorphic function f on B such that $\left| f(z) - \frac{1}{z} \right| < 1$ for all $z \in B$ with $|z| = 1$.

Problem 9. Let f be an entire function. Suppose that the set $\{ f(z) : |z| \geq 1 \}$ is not dense in $\mathbb{C}$. Prove that f is a polynomial.