Algebra Qualifying Exam Fall 2020 Solutions

1. Let $a \in \mathbb{C}$ be an arbitrary scalar. Let A_a be the (non-commutative) $\mathbb{C}$-algebra generated by the symbols s, t, modulo the relations

$$s^2 = s + 2, \tag{1a}$$

$$t^2 = t + a, \tag{1b}$$

$$sts = tst. \tag{1c}$$

a) If $a \notin \{2, 0\}$, prove that $A_a = 0$. Prove that A_2 is six-dimensional, and find a basis.

Solution: This is an application of the Bergman diamond lemma. Order the generators $t < s$. Let us place the deglex order on monomials, which satisfies the DCC. With respect to this order, the relations can be rephrased as reduction rules

$$ss \mapsto s + 2,$$

$$tt \mapsto t + a,$$

$$sts \mapsto tst.$$

The input words are $\{ss, tt, sts\}$, and a monomial is irreducible (doesn't have an input word as a subword) if and only if it lies in the set $\mathbb{B} = \{1, s, t, st, ts, tst\}$ (because it must alternate between s and t, and not contain sts). Irreducible monomials form a basis if and only if the ambiguities are resolvable, by the Bergman diamond lemma. The ambiguities are all overlap ambiguities:

$$sss, ttt, ssts, stss, ststs.$$

The two resolutions of sss are both $ss + 2s$, and similarly with ttt.

The two resolutions of $ssts$ are

$$ssts \mapsto sts + 2ts \mapsto tst + 2ts,$$

$$ssts \mapsto stst \mapsto tstt \mapsto tst + ats.$$

Thus this ambiguity is resolvable if and only if $a = 2$. If $a \neq 2$, then $tst + 2ts = ssts = tst + ats$ so that $ts(2 - a) = 0$ so that $ts = 0$. Right multiplying by s we get $ts + 2t = 0$, so that $t = 0$. Right multiplying by t we get $a = 0$, so if $a \neq 0$ then the whole algebra is zero.

Note: if $a = 0$ then $t = 0$, and the presentation just gives the two-dimensional algebra $\mathbb{C}[s]/(s^2 - s - 2)$.

The resolution of $stss$ is the same, after applying a horizontal flip to all words.

The final ambiguity $ststs$ resolves as follows:

$$ststs \mapsto tstts \mapsto tsts + atss \mapsto ttst + ats + 2at \mapsto tst + ast + ats + 2at.$$

$$ststs \mapsto sttst \mapsto stst + asst \mapsto tstt + ast + 2at \mapsto tst + ats + ast + 2at.$$

So this ambiguity is resolvable, regardless of the value of a.

If you didn't know BDL: Many people still checked that $stst$ can be resolved two ways, and deduced that $(a - 2)st = 0$. Then you can multiply both sides by t, and so forth.

If you didn't know BDL: Let us prove that $\{1, s, t, st, ts, tst\}$ is a basis of A_2. We have an action of A_2 on M given below. We also have two one-dimensional representations where s and t both act by eigenvalue 2, or both by eigenvalue -1. Taking the direct sum of these representations, we get a map from A_2 to 4×4 matrices, and one can verify that the images of $\{1, s, t, st, ts, tst\}$ are linearly independent.

To prove that this set spans, one can take any monomial in $\{s, t\}$, and prove it lies in the span, using induction on the length of the monomial. If there are any repetitions ss or tt we can apply a relation and use induction. If there are no repetitions, but sts is a subword, we can replace it with tst, and this will create a repetition unless the word has length 3, in which case we get the word tst. Any non-repeating word of length 4 or more will have sts as a subword.

b) The algebra A_2 acts on $M = \mathbb{C}^2$ by the formulas

$$s \mapsto \begin{pmatrix} 2 & \sqrt{2} \\ 0 & -1 \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} -1 & 0 \\ \sqrt{2} & 2 \end{pmatrix}. \tag{2}$$

Prove that this A_2-module M is irreducible.

Solution: If M has a nontrivial submodule, it must be one-dimensional, and both s and t must act by a scalar. So M must be spanned by a simultaneous eigenvector for both s and t. The 2-eigenspace of s is spanned by $(1, 0)$ which is not a t-eigenvector. The -1-eigenspace of s is spanned by $(\sqrt{2}, -3)$, which is not a t-eigenvector.

c) Prove that any irreducible A_2-module is at most three-dimensional.

Solution: Since A_2 is finite-dimensional over $\mathbb{C}$, any irreducible module is finite-dimensional. Let L be an irreducible A_2-submodule. The operator s must have an eigenvalue (since we work over $\mathbb{C}$), so let us choose an s-eigenvector x with eigenvalue λ. Then the span of $\{x, tx, stx\}$ is a submodule of L, since it is preserved by both s and t. Acting by s we get: $s(x) = \lambda x$ and $s(tx) = stx$ and $s(stx) = stx + 2tx$. Acting by t we get: $t(x) = tx$, $t(tx) = tx + 2tx$, and $t(stx) = tstx = stsx = \lambda stx$. Since L is irreducible, L is spanned by $\{x, tx, stx\}$, and is at most 3-dimensional.

Note: No one got this, only one person attempted it.

2. Let L be a linear endomorphism of a finite-dimensional $\mathbb{F}$ vector space V. Suppose that L has characteristic polynomial

$$p = (x+1)^4(x-3)^2(x^2+1)^2 = x^{10} - 2x^9 - 7x^8 + 14x^6 + 36x^5 + 62x^4 + 64x^3 + 49x^2 + 30x + 9$$

and minimal polynomial

$$q = (x+1)(x-3)^2(x^2+1)^2 = x^7 - 5x^6 + 5x^5 - x^4 + 7x^3 + 13x^2 + 3x + 9.$$

OOPS! We forgot to state that $\mathbb{F}$ has characteristic not equal to 2! In characteristic 2 this would be $p = (x+1)^{10}$ and $q = (x+1)^7$. This would lead to multiple possible invariant factors (because many more things divide q). Parts (c) and (d) can clearly not both be true (the proofs using CRT are clearly incorrect because we don't use distinct irreducible factors).

a) When $\mathbb{F} = \mathbb{C}$, find the matrix for L in Jordan canonical form.

Solution: For space on this page, I'll use words. A block diagonal matrix, with four blocks of -1, one block of $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$, one block of $\begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}$, and one block of $\begin{pmatrix} -i & 1 \\ 0 & -i \end{pmatrix}$.

b) What are the invariant factors of $xI - L$? Find the matrix for L in rational canonical form.

Solution: The last invariant factor of $\det(xI - L)$ is q, and the product of the invariant factors is p, so the rest of them divide $(x+1)^3$. Since they also divide q, the nontrivial ones must each be $x+1$. So the invariant factors are $1 \mid \cdots \mid 1 \mid (x+1) \mid (x+1) \mid (x+1) \mid q$ (the total number of them is 10, since it is a 10×10 matrix). The rational canonical form of L is the block diagonal matrix of the companion matrices of the invariant factors. So there are three blocks of (-1), and one block which is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -9 \\ 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & -13 \\ 0 & 0 & 1 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & +5 \end{pmatrix}.$$

c) Prove that there is a polynomial $f(x)$ such that $f(L)$ projects from V to the (-1)-eigenspace of L. (You need not find it explicitly.)

Solution: We can consider V as a module over $\mathbb{F}[x]$ where x acts by L. This action factors through the quotient algebra $A = \mathbb{F}[x]/q$. By the Chinese Remainder Theorem,

$$A \cong \mathbb{F}[x]/(x+1) \times \mathbb{F}[x]/(x-3)^2 \times \mathbb{F}[x]/(x^2+1)^2.$$

Thus there is an idempotent (the element $(1,0,0)$ in this product) which projects to the modules for which $x+1$ acts as zero. This idempotent is the image in A of some polynomial $f \in \mathbb{F}[x]$, and $f(L)$ will do the trick.

Common error: Trying to work with the CRT result on $\mathbb{F}[x]/(p)$ instead of $\mathbb{F}[x]/(q)$.

d) Prove that there is no polynomial $g(x)$ such that $g(L)$ projects from V to the 3-eigenspace of L.

Solution: Again we consider V as a module over A. The generalized 3-eigenspace is a direct summand of V, the image of the idempotent $(0,1,0)$. However, the 3-eigenspace is a submodule which is not a summand (just as the ideal $(x-3)$ is a submodule of $\mathbb{F}[x]/(x-3)^2$ but not a summand). Thus there is no idempotent in $\mathrm{End}_A(V)$ which projects to V. Since A is commutative, the image of A inside $\mathrm{End}_{\mathbb{F}}(V)$ (and hence the image of $\mathbb{F}[x]$) is contained inside $\mathrm{End}_A(V)$.

3. This problem will classify all finite groups of order $363 = 3 \cdot 11^2$. Let C_k denote the cyclic group of order k.

i) Prove that any group G of this order is a semi-direct product.

Solution: By Sylow's theorem, the number of 11-Sylow subgroups divides 3 and is equal to 1 modulo 11. The only solution is $n_{11} = 1$, so that the Sylow 11-subgroup P is normal. Let Q be a Sylow 3-subgroup. Since $|P|$ and $|Q|$ are relatively prime we have $P \cap Q = 1$ and $|PQ| = |P||Q| = |G|$. So $G \cong Q \rtimes P$.

ii) How many elements does $\mathrm{Aut}(C_{11} \times C_{11}) = GL_2(\mathbb{F}_{11})$ have? How many elements does $\mathrm{Aut}(C_{11^2})$ have? Justify your answer.

Solution: The number of elements in $GL_2(\mathbb{F}_q)$ is $(q^2-1)(q^2-q)$, since this is the number of ordered bases in a 2-dimensional space (any nonzero vector works for the first column, any vector not in the span of the first works for the second column). So $GL_2(\mathbb{F}_{11})$ has $120 * 110 = 2^4 \cdot 3 \cdot 5^2 \cdot 11$ elements. Meanwhile, $\mathrm{Aut}(C_{121}) = \mathbb{Z}/(121\mathbb{Z})^\times$, and any element of $\mathbb{Z}/121\mathbb{Z}$ which is relatively prime to 11 will be a unit in this ring, whence the size is $121 - 11 = 110 = 2 \cdot 5 \cdot 11$. (This is also the Euler function $\varphi(121)$.)

iii) How many conjugacy classes are there of matrices $A \in GL_2(\mathbb{F}_{11})$ with $A^3 = I$? Give one matrix from each conjugacy class. (Hint: Don't use Sylow theory, use some other result.)

Solution: We use rational canonical form (this is NOT called Jordan normal form). Let p be the minimal polynomial of A. Since p divides $x^3 - 1$ and has degree at most 2, either $p = (x-1)$ or $p = x^2 + x + 1$ (note: the latter is irreducible in $\mathbb{F}_{11}$ as one can check by direct inspection, it has no roots). If $p = (x-1)$ then $A = I$. If $p = x^2 + x + 1$ then A is similar to the companion matrix of p, which is $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. Thus there are two conjugacy classes.

iv) List the isomorphism classes of groups of order 363, and justify your answer.

Solution: The Sylow subgroup P is either $C_{11} \times C_{11}$ or C_{121}, and Q is C_3. If the semi-direct product is direct, this leads to two abelian groups. Any homomorphism $Q \to \mathrm{Aut}(C_{121})$ is trivial, since the size of $\mathrm{Aut}(C_{121})$ is relatively prime to 3. So there is no nonabelian semidirect product $Q \rtimes (C_{121})$. For the rest of this problem we assume $P = C_{11} \times C_{11}$.

The isomorphism class of a semidirect product determined by a map $\varphi\colon Q \to \mathrm{Aut}(P)$ is unchanged by conjugation in $\mathrm{Aut}(P)$ or by precomposition with an automorphism of Q. The generator of $Q = C_3$ is sent by φ to a matrix $A \in GL_2(\mathbb{F}_{11})$ for which $A^3 = I$. If A is nontrivial then it has order 3. Up to conjugation, $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. Thus there is one isomorphism class of nontrivial semi-direct product $C_3 \rtimes P$.

Note: Do NOT say that the isomorphism class of the semidirect product given by φ is DETERMINED up to conjugation on the target. This is false. There's also automorphisms of Q, and there's also other surprising isomorphisms possible.

The two abelian groups are nonisomorphic by the fundamental theorem of abelian groups, and the third group is nonabelian so is non-isomorphic to the other two. *Note:* Though it is easy in this case, you must still at least address the question of whether the groups you constructed are (secretly) isomorphic.

4.

i) Give a definition (or any equivalent description) of the Jacobson radical $J(R)$ of a ring R, and use it to prove that any nilpotent element in a commutative algebra is inside its Jacobson radical.

Solution 1: The Jacobson radical is the intersection of all the maximal left ideals. In a commutative algebra, any prime ideal contains zero, so it contains every nilpotent element. Hence any nilpotent is in the intersection of all the prime ideals, whence all the maximal (left) ideals.

Solution 2: The Jacobson radical equals $\{x \in R \mid 1 + fx \text{ is a unit for all } f \in R\}$. If x is nilpotent in a commutative algebra then so is fx for all $f \in R$. For any element n with $n^N = 0$, $1 + n$ is invertible, since

$$(1+n)(1 - n + n^2 - n^3 + \cdots \pm n^{N-1}) = 1 \pm n^N = 1.$$

Solution 3: The Jacobson radical is the intersection of the annihilators of all simple left modules. If x is nilpotent and V is simple, then (since R is commutative) x gives an endomorphism of V. Thus xV is a submodule, which is either 0 or V. The latter is impossible since if $xV = V$ then $0 = x^n V = V$ for some n. Thus $xV = 0$ and x annihilates V. (Or do effectively the same argument with Schur's lemma.)

ii) Give an example of a nilpotent element in a non-commutative algebra which is not inside its Jacobson radical.

Solution: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C})$.

iii) Give an example of a homomorphism $f\colon C \to D$ of commutative $\mathbb{C}$-algebras such that $f^{-1}(J(D)) \not\subset J(C)$. You need not prove your assertion, but you should identify what $J(C)$ and $J(D)$ are.

Solution: $C = \mathbb{C}[x]$ and $J(C) = 0$. $D = \mathbb{C}[x]/(x^2)$ and $J(D) = (x)$. Clearly $f^{-1}(J(D)) = (x)$. Lots of people also did $D = \mathbb{C}[[x]]$ or $\mathbb{C}[x]_{(x)}$.

iv) Give an example of a homomorphism $f\colon A \to B$ of commutative $\mathbb{C}$-algebras such that $f(J(A)) \not\subset J(B)$. You need not prove your assertion, but you should identify what $J(A)$ and $J(B)$ are.

Solution: $A = \mathbb{C}[[x]]$ and $J(A) = (x)$. $B = \mathbb{C}((x))$ and $J(B) = 0$. Lots of people did similarly: $A = \mathbb{C}[x]_{(x)}$ and $B = \mathbb{C}(x)$.

v) Suppose R is Noetherian, and let $J = J(R)$. If $J^n = J^{n+1}$ for some n, prove that $J^n = 0$ (note: J^n is the ideal $J \cdot J \cdot \ldots \cdot J$).

Solution: J^n is an ideal and hence is a finitely-generated R-module by Noetherianity. Moreover, $J \cdot J^n = J^{n+1} = J^n$. Thus by Nakayama's lemma (which states that for a f.g. module V, $JV = V$ implies $V = 0$), $J^n = 0$.

vi) Let $f\colon E \to F$ be a ring homomorphism, and suppose that E is left Noetherian and Artinian, and F is commutative. Prove that $f(J(E)) \subset J(F)$.

Solution: If E is Noetherian and Artinian then $\cdots \subset J^n \subset J^{n-1} \subset \cdots \subset J = J(E)$ stabilizes, so $J^n = J^{n+1}$ for some n. Consequently, $J^n = 0$ for some n, and any element of $J(E)$ is nilpotent. Consequently $f(J(E))$ consists of nilpotent elements, which lie inside $J(F)$ by part (a).

5. Let $R = \mathbb{F}[X]$ be the coordinate ring of an algebraic set, over an algebraically closed field $\mathbb{F}$. If $I, J \subset R$ are two ideals, we define $(I : J) = \{r \in R \mid rJ \subset I\}$. It is easy to prove that $(I : J)$ is an ideal.

i) Prove that $\mathcal{V}(I : J)$ contains $\mathcal{V}(I) \setminus \mathcal{V}(J)$.

Solution: Suppose that $r \in (I : J)$ and $x \in \mathcal{V}(I) \setminus \mathcal{V}(J)$. Since $x \notin \mathcal{V}(J)$, there exists some $f \in J$ such that $f(x) \neq 0$. However, $rf \in I$ and $x \in \mathcal{V}(I)$ so that $r(x)f(x) = 0$. Hence $r(x) = 0$.

ii) Suppose that I is a radical ideal. Prove that $\mathcal{V}(I : J)$ is the closure of $\mathcal{V}(I) \setminus \mathcal{V}(J)$.

Solution: By part (a) we know that the $\mathcal{V}(I : J)$ contains the closure, because it is itself closed. Conversely, to show that $\mathcal{V}(I : J)$ is contained in the closure, we apply $\mathcal{Z}$ (taking the zero ideal of a set) and deduce the equivalent statement that $(I : J)$ contains those functions which kill $\mathcal{V}(I) \setminus \mathcal{V}(J)$. (Here we use that $\mathcal{Z}$ is order-reversing, and that $\mathcal{V}(\mathcal{Z}(U)) = \overline{U}$ for any set $U \subset X$.) let $r \in R$ be a polynomial which kills x for any $x \in \mathcal{V}(I) \setminus \mathcal{V}(J)$. Let $f \in J$ be arbitrary. Now let $y \in \mathcal{V}(I)$ be arbitrary. If $y \in \mathcal{V}(J)$ then $f(y) = 0$, and if $y \notin \mathcal{V}(J)$ then $r(y) = 0$. Thus $r(y)f(y) = 0$. Hence rf kills anything in $\mathcal{V}(I)$, when $rf \in \mathcal{Z}(\mathcal{V}(I)) = I$ (here we use I is radical, and the Nulstellensatz). Since f was arbitrary, $rJ \subset I$, so $r \in (I : J)$.

Note: There were some erroneous attempts at this problem using the irreducible components of $\mathcal{V}(I)$. This is a reasonable idea. Suppose I is a prime ideal so that $\mathcal{V}(I)$ is irreducible. Then one can easily show (see part (d) solution 2) that either

- $J \not\subset I$ so that $\mathcal{V}(J) \cap \mathcal{V}(I)$ is a proper closed subset, and $(I : J) = I$. Since the closure of a nonempty open set in an irreducible closed set is everything, this gives you what you want. Or,
- $J \subset I$ so that $\mathcal{V}(I) \subset \mathcal{V}(J)$ and $\mathcal{V}(I : J) = \emptyset$. This also works.

So there is merit to this path, if correctly argued.

iii) It is fairly easy to prove for ideals I, J, K that $(I : J + K) = (I : J) \cap (I : K)$. When I is a radical ideal, what is the corresponding topological statement about closed sets in the Zariski topology? (You need not prove this topological statement.)

Solution: Note that $\mathcal{V}(J + K) = \mathcal{V}(J) \cap \mathcal{V}(K)$, and $\mathcal{V}((I : J) \cap (I : K)) = \mathcal{V}(I : J) \cup \mathcal{V}(I : K)$. Thus we are stating that

$$\overline{\mathcal{V}(I) \setminus (\mathcal{V}(J) \cap \mathcal{V}(K))} = \overline{(\mathcal{V}(I) \setminus \mathcal{V}(J))} \cup \overline{(\mathcal{V}(I) \setminus \mathcal{V}(K))}.$$

Aside: Without the closures, this is an obvious set-theoretic statement. It is also clear in topology that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

iv) Let $I \subset \mathbb{F}[x, y]$ be the ideal $I = (y - x^2)$. Does there exist an ideal J such that $\mathbb{F}[x, y]/(I : J)$ is finite-dimensional but nonzero (or equivalently, $\mathcal{V}(I : J)$ is a finite non-empty set of points)? If so, find such a J. If not, prove it can't happen.

Solution: It's not possible. Suppose that $\mathcal{V}(I : J)$ is a finite non-empty set of points. If this is the closure of another set, that other set must also be a finite non-empty set of points. So $\mathcal{V}(I) \setminus \mathcal{V}(J)$ is finite and nonempty. But this means that $\mathcal{V}(I) \cap \mathcal{V}(J)$ is a cofinite set inside the parabola $\mathcal{V}(I)$, and is also closed. The Zariski topology on $\mathcal{V}(I)$ is the same as that on $\mathbb{A}^1$, and in particular, no cofinite set is closed, a contradiction.

Solution 2: It's not possible. Note that I is prime. If $J \subset I$ then $(I : J) = \mathbb{F}[x, y]$ and the quotient is zero. If $J \not\subset I$ then for any $j \in J$, $j \notin I$ and any $r \in (I : J)$ we have $rj \in I$, whence $r \in I$ because I is prime. Thus $(I : J) \subset I$, and since $I \subset (I : J)$ we have $(I : J) = I$. Then $\mathcal{V}((I : J))$ is infinite.

6. This exercise focuses on the complex representation theory of the finite group $G = GL_2(\mathbb{F}_3)$. Here is its character table: for each conjugacy class we give a representative matrix, and indicate the order of that matrix. Here, I represents the identity matrix.

C	I	$-I$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
$\lvert C\rvert$	1	1	α	8	8	6	β	γ
order	1	2	2	3	6	4	8	8
$\mathbb{1}$	1	1	1	1	1	1	1	1
det	1	1	-1	1	1	1	-1	-1
X	3	3	1	0	0	-1	-1	-1
Y	3	3	-1					
Z_1								
Z_2								
Z_3								
W								

You may freely use the fact that the commutator subgroup $[G, G]$ is equal to $SL_2(\mathbb{F}_3)$, which is the kernel of the determinant map $\det\colon G \to \{\pm 1\}$.

i) The group G acts on the set $P = \mathbb{P}^1_{\mathbb{F}_3}$ of lines inside $\mathbb{F}_3^2$. Prove that the complement to the trivial representation inside the linearization $\mathbb{C}P$ is irreducible and nontrivial.

Solution 1: If P is a G-set then $\mathrm{End}_G(\mathbb{C}P) \cong \mathbb{C}(P \times P)^G$ as vector spaces, and has dimension equal to the number of orbits of G on $P \times P$. Clearly the diagonal of $P \times P$ is an orbit. If L_1 and L_2 are two distinct lines in P, then choosing a vector in each line gives a basis of $\mathbb{F}_3^2$. Since G can send any basis to any other basis, it can send any two distinct lines to any two other distinct lines. Hence $P \times P \setminus (\Delta P)$ is a second orbit, and $\mathrm{End}_G(\mathbb{C}P)$ is two-dimensional. Since it is also semisimple, $\mathrm{End}_G(\mathbb{C}P) \cong \mathbb{C} \times \mathbb{C}$, and $\mathbb{C}P$ splits as a sum of two non-isomorphic irreducible representations.

Solution 2: Assuming the results of part (b), we know the character of this complement. Pairing this character with itself will give 1 if and only if the character is irreducible.

Note that α divides 24 and β and γ divide 6. This is because the size of a conjugacy class is $|G|$ divided by the size of the centralizer, and powers of the elements are contained in the centralizer, so the size divides $|G|$ divided by the order. Note that $|G| = 48$ (and more generally, $|GL_2(\mathbb{F}_q)| = (q^2 - 1)(q^2 - q)$).

The pairing of X with itself is

$$\frac{1}{48}(3^2 + 3^2 + \alpha + 6 + \beta + \gamma) = \frac{1}{48}(24 + \alpha + \beta + \gamma).$$

Thus the pairing is bigger than zero and less than 2, and is guaranteed to be an integer, so it must be 1. (And we could deduce $\alpha = 12$ and $\beta = \gamma = 6$ if we wanted.)

ii) Continuing, compute the character of this complement, and explain why it agrees with X in the table above. You need only justify the computation of any two non-identity entries in the table.

Solution: If P is a G-set then let V be the complement of the trivial representation inside $\mathbb{C}P$: one has $\chi_V(g) = |P^g| - 1$. Thus one can compute this character by computing the number of lines which are fixed by a matrix. There are a total of $q+1 = 3+1 = 4$ lines in P. A scalar matrix preserves all four lines, whence the entries 3 in the character of X. If $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ preserves the span of (a, b) then $(a, -b) = \lambda(a, b)$. The only solutions are $\lambda = 1$ and $b = 0$ (these solutions lie on one line), or $\lambda = -1$ and $a = 0$ (these solutions lie on another line), so exactly two lines are preserved by this matrix. This explains the entry 1 in the character of X. Similar computations work for the rest of the entries.

iii) Explicitly construct another three-dimensional irreducible representation Y (not isomorphic to X), and compute its character.

Solution: Tensoring with a one-dimensional representation preserves irreducibles, and multiplies characters. So $Y = X \otimes \det$ is irreducible, and has a -1 in its character as in the table above, whence it is not isomorphic to X. The rest of the character is

$$(3, 3, -1, 0, 0, -1, 1, 1)$$

iv) Compute the dimensions of the remaining irreducible representations.

Solution: The size of G is $48 = 1^2 + 1^2 + 3^2 + 3^2 + a^2 + b^2 + c^2 + d^2$. So the sum of the remaining squares is 28. All the one-dimensional representations have already appeared, since the abelianization $G/[G, G]$ has size 2. Thus $a, b, c, d \geq 2$. The only solution is $a = b = c = 2$ and $d = 4$.

WARNING: Many students did not use the abelianization argument, but wrote: a, b, c, d divide 48, which rules out $1, 1, 1, 5$ as a solution, so all that remains is $2, 2, 2, 4$. How did so many students overlook the possibility $1, 3, 3, 3$?! This baffles me... there aren't that many things to try! Anyway, $1, 3, 3, 3$ needs to be ruled out in a different way. You should really count the 1D representations as a first step.

v) Fill in the column associated to $-I$.

Solution: Since a central element acts by a scalar (which is a root of unity), the character will be equal to the dimension times a root of unity. In this case, $(-I)^2 = I$ so the root of unity is ± 1. So the entries will be $\pm 2, \pm 2, \pm 2, \pm 4$. Using column orthogonality between $-I$ and I, we see that

$$1^2 + 1^2 + 3^2 + 3^2 \pm 2^2 \pm 2^2 \pm 2^2 \pm 4^2 = 0.$$

If 4 appears with a plus sign then clearly there is no solution. So $-I$ has a character value -4, and $0 = 4 \pm 2^2 \pm 2^2 \pm 2^2$. The only way this can happen is if two of the representations have value -2 and one has value $+2$. So this column has entries $(1, 1, 3, 3, 2, -2, -2, -4)$.

Note: Without observing that $-I$ is central it is not easy to prove that its character values will be $\pm \dim$. For example, there are plenty of involutions in groups which have character value 0 on an irreducible, such as a reflection in a dihedral group acting on the reflection representation.