# Qualifying Exam in Algebra, Winter 2018

**Part I.** True or false. Justify your answer by giving a proof or counterexample. 10 points each.

1. The extension $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is normal.

**Answer:** TRUE. Let $\alpha = \sqrt{2+\sqrt{2}}$; it is a root of polynomial $(x^2-2)^2 = 2$. Other roots are $\pm\sqrt{2\pm\sqrt{2}}$; note that $\sqrt{2-\sqrt{2}}\alpha = \sqrt{2} = \alpha^2-2$, that is $\sqrt{2-\sqrt{2}} = \frac{\alpha^2-2}{\alpha}$. Thus all the roots of $(x^2-2)^2 = 2$ are contained in $\mathbb{Q}(\alpha)$, so $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a splitting field of this polynomial. Hence this is a normal extension.

2. Let $U_n(\mathbb{C})$ be the ring of upper triangular $n \times n$ matrices with entries in $\mathbb{C}$. Any irreducible $U_n(\mathbb{C})-$module is one dimensional over $\mathbb{C}$.

**Answer:** TRUE. We have a homomorphism $U_n(\mathbb{C}) \to \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ sending a matrix to its diagonal. The kernel of this homomorphism consists of strictly upper triangular matrices, so it is nilpotent and is contained in the Jacobson radical of $U_n(\mathbb{C})$ (in fact its coincides with the Jacobson radical). Since the Jacobson radical acts by zero on an irreducible module we see that any irreducible $U_n(\mathbb{C})-$module is a pullback of irreducible $\mathbb{C} \oplus \cdots \oplus \mathbb{C}-$module. It is clear that any irreducible module over the latter algebra is 1-dimensional (since this algebra is commutative or by the classification of simple modules over semisimple rings).

3. The abelian group $\mathbb{Q}/\mathbb{Z}$ is flat.

**Answer:** FALSE. Consider the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2. It is injective. If $\mathbb{Q}/\mathbb{Z}$ were flat, tensoring by $\mathbb{Q}/\mathbb{Z}$ would preserve injections, so the map $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ given by multiplication by 2 would be injective too. But for example the coset of $1/2$ goes to zero so it is not.

4. A $\mathbb{C}[x,y]-$module is semisimple if and only if its restrictions to both of the subalgebras $\mathbb{C}[x]$ and $\mathbb{C}[y]$ are semisimple.

**Answer:** TRUE. Let $M$ be a semisimple $\mathbb{C}[x,y]-$module. Then it is a direct sum of irreducible $\mathbb{C}[x,y]-$modules which are 1-dimensional over $\mathbb{C}$ (since by Nullstellensatz any maximal ideal of $\mathbb{C}[x,y]$ is of codimension 1). Thus the restriction of $M$ to any $\mathbb{C}-$subalgebra is a direct sum of 1-dimensional modules, hence semisimple.

Conversely assume the restrictions of $M$ to $\mathbb{C}[x]$ and $\mathbb{C}[y]$ are semisimple. Then $M = \sum_{a\in\mathbb{C}} M_a$ where $M_a = \{m \in M | xm = am\}$ since $M_a$ is a sum of all $\mathbb{C}[x]-$submodules of $M$ isomorphic to simple module $\mathbb{C}[x]/(x-a)$. It is clear that each $M_a$ is $\mathbb{C}[y]-$submodule of $M$. Thus $M_a$ decomposes into a sum of irreducible hence 1-dimensional $\mathbb{C}[y]-$modules. Any such summand of $M_a$ is clearly $\mathbb{C}[x,y]-$submodule of $M$. Thus $M$ is a sum of 1-dimensional hence irreducible $\mathbb{C}[x,y]-$modules, hence it is semisimple.

5. The cyclotomic polynomial $\Phi_{255}(x)$ reduced modulo 2 is irreducible as an element of $\mathbb{F}_2[x]$.

**Answer:** FALSE. The polynomial $\Phi_{255}(x)$ is a divisor of $x^{255}-1$ (both over $\mathbb{Z}$ and over $\mathbb{F}_2$). Thus any root $\alpha$ of $\Phi_{255}(x)$ satisfies $\alpha^{255}=1$ whence $\alpha^{256}=\alpha$. Thus $\alpha$ is contained in $\mathbb{F}_{256}$ which is the splitting field of $x^{256}-x$ over $\mathbb{F}_2$. Since $[\mathbb{F}_{256}:\mathbb{F}_2]=8$, the degree of the minimal polynomial of $\alpha$ over $\mathbb{F}_2$ is $\leq 8$. Thus $\Phi_{255}(x)$ is not irreducible as its degree $\phi(255)=2\cdot 4\cdot 16=128>8$.

**Part II.** Longer problems. 10 points each.

1. Describe all proper subgroups of the symmetric groups $S_n$ of order strictly more than $(n-1)!$.

**Solution:** Let $H\subset S_n$ be a proper subgroup with $|H|>(n-1)!$. The group $S_n$ then acts transitively (hence nontrivially) on the set of cosets $S_n/H$ of size $m=|S_n:H|<n$. Thus we have a nontrivial homomorphism $S_n\to S_m$ and its restriction to the alternating group $A_n\to S_m$. The latter homomorphism must be trivial for $n\geq 5$ since the alternating group is simple and $|A_n|=\frac{1}{2}n!>m!=|S_m|$. Thus the action factors through $S_n/A_n=\mathbb{Z}/2\mathbb{Z}$ and its orbit $S_n/H$ is of size $\leq 2$. Thus $H=A_n$ since $A_n$ is a unique subgroup of index 2 in $S_n$.

It remains to consider the cases when $n\leq 4$. The cases $n=1,2,3$ are trivial with a unique possibility $H=A_3\subset S_3$. In the case $n=4$ the index of $H$ must be 2 or 3; if the index is 2 then the subgroup is $A_4\subset S_4$. If the index is 3 then $|H|=8$ and $H$ is Sylow 2-subgroup of $S_4$. There are precisely 3 such subgroups.

**Answer:** Such subgroup is either the alternating group $A_n\subset S_n$ for $n\geq 3$ or one of three Sylow 2-subgroups of $S_4$.

2. Let $G$ be a finite group and let $H\subset G$ be a subgroup. Let $g\in G$ be an element such that no conjugate of $g$ is contained in $H$. Prove that for any finite dimensional $H-$module $V$ (over an arbitrary field) the trace of $g$ in $\mathrm{Ind}_H^G V$ is zero.

**Solution:** Let $g_1,\dots,g_n$ be $G/H$ coset representatives. Let $v_1,\dots,v_m$ be a basis for $V$. Then $g_i\otimes v_j$ is a basis for the induced module. To compute the trace of $g$, act on this basis. Say $gg_i=g_kh$ for $h\in H$. Then $g(g_i\otimes v_j)=g_k\otimes hv_j$. The diagonal entry of the matrix of $g$ in the basis above is the coefficient of $g_i\otimes v_j$ in the expansion of $g(g_i\otimes v_j)$. Thus to give a non-zero contribution to the trace, we must have that $k=i$. But then $gg_i=g_ih$ contradicting the hypothesis on $g$.

3. For a partially ordered set $(X,\leq)$, let $\mathcal{C}_X$ be the corresponding category: the objects of $\mathcal{C}_X$ are the elements of X and there is a unique morphism $\theta:x\mapsto y$ if and only if $x\leq y$. For an order preserving map $f:X\to Y$, let $F_f:\mathcal{C}_X\to\mathcal{C}_Y$ be the corresponding functor. Viewing $\mathbb{Z}$ and $\mathbb{R}$ as partially ordered sets via the usual ordering $\leq$, the obvious embedding $i:\mathbb{Z}\to\mathbb{R}$ is an order preserving map. Find the right and left adjoints of the functor $F_i:\mathcal{C}_\mathbb{Z}\to\mathcal{C}_\mathbb{R}$, justifying your answer carefully.

**Solution:** Let $G:\mathcal{C}_\mathbb{R}\to\mathcal{C}_\mathbb{Z}$ be the left adjoint functor of $F_i$. Thus we must have a bijection $\mathrm{Hom}(Gx,m)\leftrightarrow\mathrm{Hom}(x,F_im)$ for all $x\in\mathbb{R},m\in\mathbb{Z}$. Thus

$$Gx\leq m\Leftrightarrow\mathrm{Hom}(Gx,m)\neq\emptyset\Leftrightarrow\mathrm{Hom}(x,F_im)\neq\emptyset\Leftrightarrow x\leq m\Leftrightarrow\lceil x\rceil\leq m,$$

where $\lceil\ \rceil:\mathbb{R}\to\mathbb{Z}$ is the ceiling function. Notice that this function is order preserving. Thus it is natural to expect that $G=F_{\lceil\ \rceil}$. This is indeed the case:

we have a unique bijection $\mathrm{Hom}(F_{\lceil\ \rceil}x, m) \leftrightarrow \mathrm{Hom}(x, F_i m)$ since both sets have the same cardinality which is $\leq 1$. This bijection is natural in both variables as all the Hom$-$sets in the naturality diagram are of cardinality $\leq 1$, so it must be commutative.

Similarly, the right adjoint functor of $F_i$ is $F_{\lfloor\ \rfloor}$ where $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ is the floor function. Here is a cheap way to see this: observe that the map $x \mapsto -x$ gives an equivalence to opposite categories (coming from opposite posets) and note that $\lfloor x \rfloor = -\lceil -x \rceil$.

4. Let $I \lhd \mathbb{C}[x_1, \dots, x_n]$ be an ideal such that $\sqrt{I}$ is maximal. Prove that $\mathbb{C}[x_1, \dots, x_n]/I$ is finite dimensional over $\mathbb{C}$.

**Solution:** Let $\sqrt{I} = (x_1 - c_1, \dots, x_n - c_n)$ for $(c_1, \dots, c_n) \in \mathbb{C}^n$. The monomials $\prod_{i=1}^n (x_i - c_i)^{m_i}$ with $m_i \in \mathbb{Z}_{\geq 0}$ form a basis of $\mathbb{C}[x_1, \dots, x_n]$ (e.g. apply the automorphism $x_i \mapsto x_i - c_i$ to the standard monomial basis of $\mathbb{C}[x_1, \dots, x_n]$. By definition of $\sqrt{I}$, for any $i = 1, \dots, n$ there is $n_i \in \mathbb{Z}_{>0}$ such that $(x_i - c_i)^{n_i} \in I$. Thus the monomials $\prod_{i=1}^n (x_i - c_i)^{m_i}$ with $0 \leq m_i < n_i$ for all $i$ span $\mathbb{C}[x_1, \dots, x_n]/I$. Hence $\mathbb{C}[x_1, \dots, x_n]/I$ is finite dimensional of dimension $\leq \prod_{i=1}^n n_i$.

5. Let $V$ be a finite dimensional vector space over a field $F$, and let $f : V \to V$ be a linear transformation. Prove that $2\mathrm{tr}(S^2 f) = \mathrm{tr}(f)^2 + \mathrm{tr}(f^2)$.

**Solution:** As the extension of the field does not change the traces we can and will assume that $F$ is algebraically closed. Pick a basis $v_1, .., v_n$ with respect to which $f$ is upper triangular with $\lambda_1, \dots, \lambda_n$ on the diagonal (e.g. Jordan normal form basis would work). Then $v_i v_j$ with $i \leq j$ is a basis for $S^2 V$ and the matrix of $S^2 f$ has $\lambda_i \lambda_j$ on its diagonal. We deduce that $2\mathrm{tr}(S^2 f) = 2 \sum_{i \leq j} \lambda_i \lambda_j$. On the other hand $(\mathrm{tr}(f))^2 + \mathrm{tr}(f^2) = \sum \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j + \sum \lambda_i^2$. The result follows.