Qualifying Exam in Algebra, Winter 2018

Part I. True or false. Justify your answer by giving a proof or counterexample. 10 points each.

1. The extension $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is normal.
2. Let $U_n(\mathbb{C})$ be the ring of upper triangular $n \times n$ matrices with entries in $\mathbb{C}$. Any irreducible $U_n(\mathbb{C})-$module is one dimensional over $\mathbb{C}$.
3. The abelian group $\mathbb{Q}/\mathbb{Z}$ is flat.
4. A $\mathbb{C}[x,y]-$module is semisimple if and only if its restrictions to both of the subalgebras $\mathbb{C}[x]$ and $\mathbb{C}[y]$ are semisimple.
5. The cyclotomic polynomial $\Phi_{255}(x)$ reduced modulo 2 is irreducible as an element of $\mathbb{F}_2[x]$.

Part II. Longer problems. 10 points each.

1. Describe all proper subgroups of the symmetric groups S_n of order strictly more than $(n-1)!$.
2. Let G be a finite group and let $H \subset G$ be a subgroup. Let $g \in G$ be an element such that no conjugate of g is contained in H. Prove that for any finite dimensional $H-$module V (over an arbitrary field) the trace of g in $\operatorname{Ind}_H^G V$ is zero.
3. For a partially ordered set $(X, \leq)$, let $\mathcal{C}_X$ be the corresponding category: the objects of $\mathcal{C}_X$ are the elements of X and there is a unique morphism $\theta : x \mapsto y$ if and only if $x \leq y$. For an order preserving map $f : X \to Y$, let $F_f : \mathcal{C}_X \to \mathcal{C}_Y$ be the corresponding functor. Viewing $\mathbb{Z}$ and $\mathbb{R}$ as partially ordered sets via the usual ordering $\leq$, the obvious embedding $i : \mathbb{Z} \to \mathbb{R}$ is an order preserving map. Find the right and left adjoints of the functor $F_i : \mathcal{C}_{\mathbb{Z}} \to \mathcal{C}_{\mathbb{R}}$, justifying your answer carefully.
4. Let $I \lhd \mathbb{C}[x_1, \ldots, x_n]$ be an ideal such that $\sqrt{I}$ is maximal. Prove that $\mathbb{C}[x_1, \ldots, x_n]/I$ is finite dimensional over $\mathbb{C}$.
5. Let V be a finite dimensional vector space over a field F, and let $f : V \to V$ be a linear transformation. Prove that $2\text{tr}(S^2 f) = \text{tr}(f)^2 + \text{tr}(f^2)$.