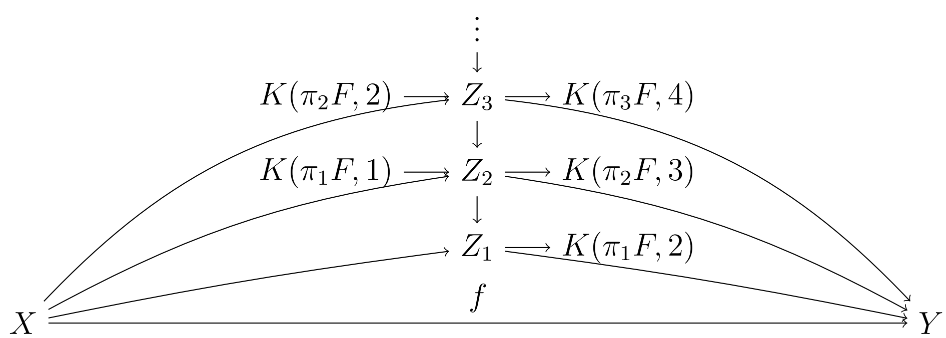
UO TOPOLOGY QUALIFYING EXAM
FALL 2019

Precise statements of selected definitions and theorems

These may or may not be useful to you in solving the problems on the next page.

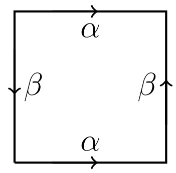
- A pair (X, A) is *n-connected* if each path component of X contains a unique path component of A and $\pi_i(X, A, x_0) = 0$ for all $i \leq n$ and $x_0 \in A$.
- *Hurewicz Theorem.* If X is $(n-1)$-connected, $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i < n$ and the Hurewicz map $\pi_n(X) \to H_n(X)$ is an isomorphism. If (X, A) is $(n-1)$-connected, $n \geq 2$, with A simply connected and non-empty then $H_i(X, A) = 0$ for $i < n$ and the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ is an isomorphism. More generally, for A not simply connected, $\pi_n(X, A) \to H_n(X, A)$ is surjective with kernel the normal subgroup generated by $\{\gamma \cdot f - \gamma\}$ for $\gamma \in \pi_1(A)$ and $f \in \pi_n(X, A)$ and $H_i(X, A) = 0$ for $i < n$.
- *Excision theorem for homotopy groups.* Let (A, C) and (B, C) be CW pairs and $X = A \cup B$. If (A, C) is m-connected and (B, C) is n-connected then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.
- *Frudenthal Suspension Theorem.* If X is an $(n-1)$-connected CW complex then the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.
- A map $f : X \to Y$ has a Moore-Postnikov tower of principal fibrations



if and only if $\pi_1(X)$ acts trivially on $\pi_n(\mathrm{Cyl}(f), X)$ for all $n > 1$ or, equivalently, on $\pi_n(F)$ (the homotopy fiber of f) for all $n \geq 1$.

Problems

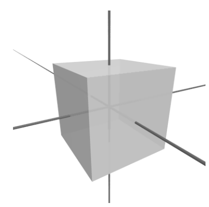
(1) Recall that the Klein bottle K can be described as the following identification space:



Show that the Klein bottle retracts onto one of the circles α, β but not the onto the other.

(2) Let $M(3,1)$ be the result of attaching a 2-cell to S^1 by the map $z \mapsto z^3$. Describe explicitly, with proof, all connected covering spaces of $M(3,1) \times \mathbb{R}P^2$. (Your answers do not have to be given with specific cell structures, but "explicit" means it should be clear they are homeomorphic to CW complexes—no path spaces.)

(3) Let X be the union of the (hollow) cube $\partial([-1,1]^3)$ and the three coordinate axes in $\mathbb{R}^3$



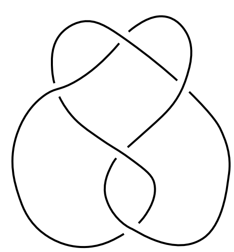
(a) Compute $\pi_1(X)$.
(b) Compute the homology groups of X.

(4) Let $\phi\colon S^2 \times S^2 \to S^2 \times S^2$ be the map $\phi(x,y) = (y,x)$. Let $T_\phi = (S^2 \times S^2 \times [0,1])/((x,y,1) \sim (y,x,0))$ be the mapping torus of ϕ. Compute the homology groups of T_ϕ.

(5) (a) Define the compactly supported cohomology groups H^i_c of a space X.
(b) Show that H^i_c is not a cohomology theory. More precisely, show that there is no cohomology theory h^* so that $h^i(X) \cong H^i_c(X)$ for all spaces X and integers i.

(6) Let $(\mathbb{R}P^2)^{2019}$ be the product of 2019 copies of $\mathbb{R}P^2$ with itself. Suppose $f\colon (\mathbb{R}P^2)^{2019} \to (\mathbb{R}P^2)^{2019}$ is a continuous. Show f has a fixed point.

(7) Consider the knot 5_2



Recall that a connected covering space $p\colon \widetilde{X} \to X$ is *normal* (sometimes called *regular*) if $p_*\pi_1(\widetilde{X}, \widetilde{x}_0)$ is a normal subgroup of $\pi_1(X, x_0)$ or, equivalently, if the deck group of $\widetilde{X}$ acts transitively on each fiber of p.

(a) I claim I have a normal covering space of $S^3 \setminus 5_2$ with deck transformation group $\mathbb{Z}/537\mathbb{Z}$. Do you believe me? Justify.
(b) Now I claim I have a normal covering space of $S^3 \setminus 5_2$ with deck transformation group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Do you believe me? Justify.

(8) Let G be a finitely generated abelian group. Show that no closed 3-manifold is a $K(G, 2)$. (Do not use the Poincaré conjecture.) (Hint: first reduce to the orientable case.)

(9) Recall that orientable k-dimensional vector bundles over X are in bijection with $[X, \mathrm{Gr}_3^+(\mathbb{R}^\infty)]$, where $\mathrm{Gr}_3^+(\mathbb{R}^\infty) = V_3(\mathbb{R}^\infty)/\,SO(3)$ is the Grassmanian of oriented 3-planes in $\mathbb{R}^\infty$. Show that $\mathrm{Gr}_3^+(\mathbb{R}^\infty)$ is simply-connected and $\pi_2(\mathrm{Gr}_3^+(\mathbb{R}^\infty)) \cong \mathbb{Z}/2\mathbb{Z}$, and compute $\pi_3(\mathrm{Gr}_3^+(\mathbb{R}^\infty))$ and $\pi_4(\mathrm{Gr}_3^+(\mathbb{R}^\infty))$. (Hint: recall that $SO(3) \cong \mathbb{R}P^3$.)

(10) Let Y be a 2-connected space and $p\colon Y \to \mathrm{Gr}_3^+(\mathbb{R}^\infty)$ a fibration so that $p_*\colon \pi_i(Y) \to \pi_i(\mathrm{Gr}_3^+(\mathbb{R}^\infty))$ is an isomorphism for $i > 2$. (That is, Y is a 2-*connected cover* of $\mathrm{Gr}_3^+(\mathbb{R}^\infty)$.) Define the (primary) obstruction in cohomology to lifting a map $f\colon X \to \mathrm{Gr}_3^+(\mathbb{R}^\infty)$ to a map $\widetilde{f}\colon X \to Y$ and give an example where the obstruction does not vanish.