(1) Recall that the Klein bottle $K$ can be described as the following identification space:

![Diagram of the Klein bottle]

Show that the Klein bottle retracts onto one of the circles $\alpha, \beta$ but not the other.

**Solution 1.** Identify the square above with $[0, 1]^2$ in the obvious way. The map $f: [0, 1]^2 \to [0, 1]/(0 \sim 1) = \alpha$ given by $f(x, y) = x$ respects the equivalence relation, hence descends to a continuous map $\bar{f}: K \to \alpha$. By definition, $\bar{f}|_{\alpha}$ is the identity map, so $\bar{f}$ is a retraction.

Next, we show that there is no retraction $r: K \to \beta$. If we let $H_1$ denote the abelianization of $\pi_1$, van Kampen’s theorem gives $H_1(K) \cong \mathbb{Z}(\alpha, \beta)/(2\beta)$ (where we are abusing notation to let $\alpha$ and $\beta$ denote the elements of $\pi_1(K)$ which go around $\alpha$ and $\beta$ once). Let $i: \beta \hookrightarrow K$ denote inclusion. If there were a retraction $r: K \to \beta$, so $r \circ i = \mathbb{1}_\beta$, then we would have

$$r_* \circ i_* = \mathbb{1}: H_1(\beta) \to H_1(\beta).$$

But $H_1(\beta) = \mathbb{Z}(\beta)$, so $i_*: H_1(\beta) \to H_1(K)$ is not injective, a contradiction.

**Solution 2 (sketch).** The same as above, except use $H_1$ computed by cellular homology or the Mayer-Vietoris theorem.

(2) Let $M(3, 1)$ be the result of attaching a 2-cell to $S^1$ by the map $z \mapsto z^3$. Describe explicitly, with proof, all connected covering spaces of $M(3, 1) \times \mathbb{R}P^2$.

**Solution.** First, recall that the covering spaces of $X \times Y$ are exactly the products of covering spaces of $X$ and covering spaces of $Y$. (One can prove this directly, or from the classification of covering spaces and the fact that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. A proof is not required for full credit for this problem.) Now, the connected covering spaces of $M(3, 1)$ are in bijection with subgroups of $\pi_1(M(3, 1)) = \mathbb{Z}/3\mathbb{Z}$, of which there are two: $\mathbb{Z}/3\mathbb{Z}$ and $\{0\}$. Similarly, covering spaces of $\mathbb{R}P^2$ are in bijection with subgroups of $\mathbb{Z}/2\mathbb{Z}$, of which the only two are $\mathbb{Z}/2\mathbb{Z}$ and $\{0\}$. Hence, there are four connected covering spaces of $M(3, 1) \times \mathbb{R}P^2$.

The two connected covering spaces of $\mathbb{R}P^2$ are $\mathbb{1}: \mathbb{R}P^2 \to \mathbb{R}P^2$ and the quotient map $S^2 \to S^2/\{\pm 1\} = \mathbb{R}P^2$.

The two connected covering spaces of $M(3, 1)$ are $\mathbb{1}: M(3, 1) \to M(3, 1)$ and another one, $f: X \to M(3, 1)$ defined as follows. Let

$$X = D^2 \times \{0, 1, 2\}/ \sim$$
where \((x, i) \sim (x, j)\) for each \(x \in \partial D^2\). Let \(q : D^2 \to M(3, 1)\) be the quotient map. Then \(f(x, j) = q(e^{2\pi j \sqrt{-1}/3} x)\). (A clear picture would also suffice here, though \(M(3, 1)\) does not embed in \(\mathbb{R}^3\).)

Now, the four connected covering spaces of \(M(3, 1) \times \mathbb{R}P^2\) are \(M(3, 1) \times \mathbb{R}P^2\), \(M(3, 1) \times S^2\), \(X \times \mathbb{R}P^2\), and \(X \times S^2\), with the obvious maps.

(3) Let \(X\) be the union of the (hollow) cube \(\partial([−1, 1]^3)\) and the three coordinate axes in \(\mathbb{R}^3\).

(a) Compute \(\pi_1(X)\).
(b) Compute the homology groups of \(X\).

**Solution.** We start by replacing \(X\) by a homotopy equivalent space where the computations are easier. First, \(X\) deformation retracts to the union of the hollow cube and the parts of the coordinate axes lying inside the cube. Call the image of this deformation retraction \(Y\). The space \(Y\) can be given the structure of a CW complex with, say:

- 0-skeleton \(\{(0, 0, 0), (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}\),
- 1-skeleton \(Y \cap \{(x, y, z) \mid xyz = 0\}\)
- 8 2-cells, around the eight vertices of the cube.

(A good picture would be a fine substitute for words here.) Let \(Z \subset Y\) be the union of:

- 5 of the 6 faces of the cube, and
- the segment from one of those five faces to \((0, 0, 0)\).

Then \(Z\) is a contractible subcomplex of \(Y\), and the space \(Y/Z\) is homeomorphic to the wedge sum of \(S^2\) and 5 circles,

\[ S^2 \vee S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1. \]

Since both \(\pi_1\) and \(H_*\) are homotopy invariants,

\[
\pi_1(X) \cong \pi_1(S^2 \vee S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1) \\
H_*(X) \cong H_*(S^2 \vee S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1).
\]

So, it is immediate from van Kampen’s theorem and cellular homology that

\[
\pi_1(X) \cong \ast_{i=1}^5 \pi_1(S^1) \cong F_5 \\
H_0(X) \cong \mathbb{Z} \\
H_1(X) \cong \bigoplus_{i=0}^5 H_1(S^1) \cong \mathbb{Z}^5 \\
H_2(X) \cong H_2(S^2) \cong \mathbb{Z} \\
H_i(X) = 0 \quad i > 2
\]

(Students do not need to spell out further details here for full credit.)

**Solution 2 (sketch).** Quotient by a different contractible subcomplex, or apply van Kampen’s theorem, and the Mayer-Vietoris sequence or cellular homology, directly to \(X\).
(4) Let $\phi: S^2 \times S^2 \to S^2 \times S^2$ be the map $\phi(x, y) = (y, x)$. Let $T_\phi = (S^2 \times S^2 \times [0, 1])/((x, y, 1) \sim (y, x, 0))$ be the mapping torus of $\phi$.

(a) Compute the homology groups of $T_\phi$.

Solution 1. Let

$$U = T_\phi \setminus (S^2 \times S^2 \times \{0\})$$
$$V = T_\phi \setminus (S^2 \times S^2 \times \{1/2\}).$$

There is an obvious homeomorphism $f: U \xrightarrow{\sim} S^2 \times S^2 \times (0, 1)$. There is also a homeomorphism $g: V \xrightarrow{\sim} S^2 \times S^2 \times (1/2, 3/2)$ defined by

$$g(p, t) = \begin{cases} (\phi^{-1}(p), t + 1) & 0 \leq t < 1/2 \\ (p, t) & 1/2 \leq t \leq 1. \end{cases}$$

For $I = (0, 1)$, $I = (1/2, 3/2)$, $I = (0, 1/2)$, or $I = (1/2, 1)$, let $p: S^2 \times S^2 \times I \to S^2 \times S^2$ denote projection. Then we have isomorphisms

$$(p \circ f)_* : H_*(U) \xrightarrow{\sim} H_*(S^2 \times S^2)$$
$$(p \circ g)_* : H_*(V) \xrightarrow{\sim} H_*(S^2 \times S^2)$$

$$((p \Pi p) \circ f)_* : H_*(U \cap V) \xrightarrow{\sim} H_*(S^2 \times S^2) \oplus H_*(S^2 \times S^2).$$

Apply the Mayer-Vietoris theorem to the cover $T_\phi = U \cup V$ and use the identifications above to obtain

$$\cdots \to H_i(U \cap V) \xrightarrow{(p \Pi p) \circ f}_* \to H_i(U) \oplus H_i(V) \xrightarrow{(p \circ f)_* \oplus (p \circ g)_*} H_i(T_\phi) \to \cdots$$

The map $\Psi$ is the unique map so that the diagram commutes. It follows from the definitions that

$$\Psi_i = \begin{pmatrix} 1 & 1 \\ (\phi^{-1})_* & 1 \end{pmatrix}.$$  

(Depending on one’s sign convention for the Mayer-Vietoris sequence, there might be minus signs in the second row.) There is a short exact sequence

$$0 \to \text{coker}(\Phi_i) \to H_i(T_\phi) \to \ker(\Phi_{i-1}) \to 0.$$  

Note that $\phi^2 = 1$, so $(\phi^{-1})_* = \phi_*$. Also, row-reducing,

$$\ker(\Phi_i) = \ker(\phi_* - 1)$$
$$\text{coker}(\Phi_i) = \text{coker}(\phi_* - 1).$$

From cellular homology (or the Küneth theorem), we have

$$H_i(S^2 \times S^2) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^2 & i = 2 \\ \mathbb{Z} & i = 4 \\ 0 & \text{else}. \end{cases}$$
Further, by considering degrees, $\phi_*$ is the identity map on $H_0$, the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $H_2$, and the identity map on $H_4$.

Hence, we have
\[
H_0(T\phi) \cong \ker((\phi_* - I): H_0(S^2 \times S^2) \to H_0(S^2 \times S^2)) \cong \mathbb{Z}
\]
\[
H_1(T\phi) \cong \ker((\phi_* - I): H_0(S^2 \times S^2) \to H_0(S^2 \times S^2)) \cong \mathbb{Z}
\]
\[
H_2(T\phi) \cong \ker((\phi_* - I): H_2(S^2 \times S^2) \to H_2(S^2 \times S^2)) \cong \ker\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cong \mathbb{Z}
\]
\[
H_3(T\phi) \cong \ker((\phi_* - I): H_2(S^2 \times S^2) \to H_2(S^2 \times S^2)) \cong \ker\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cong \mathbb{Z}
\]
\[
H_4(T\phi) \cong \ker((\phi_* - I): H_4(S^2 \times S^2) \to H_4(S^2 \times S^2)) \cong \mathbb{Z}
\]
\[
H_5(T\phi) \cong \ker((\phi_* - I): H_4(S^2 \times S^2) \to H_4(S^2 \times S^2)) \cong \mathbb{Z}
\]

**Solution 2. (sketch).** Hatcher gives a long exact sequence for the homology of a mapping torus, which we did not cover in class but which some students might know.

**Solution 3. (sketch).** It is a bit tedious, but this computation can be done using cellular homology.

(5) (a) Define the compactly supported cohomology groups $H^i_c$ of a space $X$.

**Solution.** If $K \subset L \subset X$ then $X \setminus K \supset X \setminus L$. Hence, the inclusion map of pairs $(X, X \setminus L) \hookrightarrow (X, X \setminus K)$ induces a map of relative cohomology $H^i(X, X \setminus K) \to H^i(X, X \setminus L)$. Further, if $K \subset L \subset M$ then, since the diagram of inclusions
\[
\begin{array}{ccc}
(X, X \setminus K) & \hookrightarrow & (X, X \setminus M) \\
(X, X \setminus L) & \hookrightarrow & (X, X \setminus L)
\end{array}
\]
commutes, the diagram of relative cohomologies
\[
\begin{array}{ccc}
H^i(X, X \setminus K) & \to & H^i(X, X \setminus M) \\
& \to & H^i(X, X \setminus L)
\end{array}
\]
commutes.

Hence, the groups
\[
\{H^i(X, X \setminus K)\}_{K \subset X \text{ compact}}
\]
form a directed system. The compactly supported cohomology $H^i_c(X)$ is the direct limit of this directed system.

**Solution 2 (sketch).** Alternatively, one can define
\[
C^i_c(X) = \varinjlim_{K \subset X \text{ compact}} C^i(X, X \setminus K),
\]
see that $d$ induces a map $d: C^i_c(X) \to C^{i+1}_c(X)$ and these maps form a chain complex, and define $H^i_c(X)$ to be the homology of this chain complex.
(b) Show that $H^i_c$ is not a cohomology theory. More precisely, show that there is no cohomology theory $h^*$ so that $h^i(X) \cong H^i_c(X)$ for all spaces $X$ and integers $i$.

**Solution.** If $h^*$ is a cohomology theory then the homotopy axiom implies that if $X \simeq Y$ then $h^i(X) \cong h^i(Y)$ for all $i$. For compactly supported cohomology, by definition $H^0_c(\mathbb{R}^0) = H^0(\mathbb{R}^0, \emptyset) \cong \mathbb{Z}$. On the other hand, $H^0_c(\mathbb{R}^1) = 0$: it follows from Poincaré duality that $H^0_c(\mathbb{R}^1) \cong H_1(\mathbb{R}^1) = 0$. (Alternatively, it is not hard to show directly that $H^0_c(\mathbb{R}^1) = 0$.)

**Remark.** Compactly-supported cohomology is functorial under proper maps (though not all maps), and invariant under proper homotopies.

(6) Let $(\mathbb{R}P^2)^{2019}$ be the product of 2019 copies of $\mathbb{R}P^2$ with itself. Suppose $f : (\mathbb{R}P^2)^{2019} \to (\mathbb{R}P^2)^{2019}$ is a continuous. Show $f$ has a fixed point.

**Solution.** Recall that the homology of $\mathbb{R}P^2$ is

$$H_i(\mathbb{R}P^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & i = 1 \\ 0 & \text{otherwise}. \end{cases}$$

(This follows easily, for example, from cellular homology, or from the long exact sequence for a pair or the Mayer-Vietoris sequence.) Hence, by the universal coefficient theorem,

$$H_i(\mathbb{R}P^2; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0 \\ 0 & \text{otherwise}. \end{cases}$$

Now, by the Künneth theorem,

$$H_i((\mathbb{R}P^2)^{2019}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0 \\ 0 & \text{otherwise}. \end{cases}$$

For any map $f : X \to X$, $f_* : H_0(X) \to H_0(X)$ is the identity map. Hence, for any map $f : (\mathbb{R}P^2)^{2019} \to (\mathbb{R}P^2)^{2019}$, the Lefschetz trace $\tau(f) = 1$. Hence, $f$ has a fixed point.

(7) Consider the knot $5_2$

(a) I claim I have a normal covering space of $S^3 \setminus 5_2$ with deck transformation group $\mathbb{Z}/537\mathbb{Z}$. Do you believe me? Justify.

(b) Now I claim I have a normal covering space of $S^3 \setminus 5_2$ with deck transformation group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Do you believe me? Justify.

**Solution 1.** From the classification of covering spaces, a space $X$ has a normal covering space $\tilde{X}$ with deck group $G$ if and only if $\pi_1(X)$ has a normal subgroup $H$ with $\pi_1(X)/H \cong G$. Further, if $G$ is abelian then $H$ must contain the commutator subgroup of $\pi_1(X)$, so

$$H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)] \to \pi_1(X)/H \cong G.$$
Conversely, if \( H_1(X) \) surjects onto \( G \) then \( \ker(\pi_1(X) \to H_1(X) \to G) \) corresponds to a normal covering space with deck group \( G \).

By Alexander duality, \( H_1(S^3 \setminus 5_2) \cong H^1(S^1) \cong \mathbb{Z} \). Since \( \mathbb{Z} \) surjects onto \( \mathbb{Z}/537\mathbb{Z} \), \( S^3 \setminus 5_2 \) does have a normal covering space with deck group \( \mathbb{Z}/537\mathbb{Z} \). Since \( \mathbb{Z} \) does not surject onto \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), \( S^3 \setminus 5_2 \) does not have a normal covering space with deck group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

**Solution 2 (sketch).** Students in the class have seen the Wirtinger presentation for \( \pi_1(S^3 \setminus K) \) and could substitute that for Alexander duality (though it is slightly tedious). Let \( G \) be a finitely generated abelian group. Show that no closed 3-manifold is a \( K(G, 2) \).

(Hint: reduce to the orientable case and consider the homology of a \( K(G, 2) \).)

**Solution 1.** Let \( M \) be a closed 3-manifold. If \( M \) is non-orientable then the orientation double cover of \( M \) is a nontrivial 2-fold cover, so \( \pi_1(M) \neq 0 \), so \( M \) is not a \( K(G, 2) \).

Next, if \( M \) is orientable then by Poincaré duality, \( H_1(M) \cong \mathbb{Z} \). So, it suffices to show that \( H_3(K(G, 2)) = 0 \). We can build a space \( K(G, 2) \) as follows. Start with a Moore space built from 2-cells and 3-cells. By cellular approximation (or van Kampen’s theorem), \( \pi_1(M(G, 2)) = 0 \), and by the Hurewicz theorem, \( \pi_2(M(G, 2)) \cong H_2(M(G, 2)) \cong G \). Now, attach 4-cells to \( M(G, 2) \) to kill of \( \pi_3(M(G, 2)) \), attach 5-cells to the result to kill of \( \pi_4 \), and so on. Since the resulting \( K(G, 2) \) has the same 3-skeleton as \( M(G, 2) \), \( H_3(K(G, 2)) \) is a quotient of \( H_3(M(G, 2)) = 0 \), hence vanishes. In particular, \( H_3(M) \neq H_3(K(G, 2)) \).

**Solution 2.** Suppose that \( M \) is a \( K(G, 2) \). As in Solution 1, \( M \) is orientable. By the 1-dimensional Hurewicz theorem, \( H_1(M) = 0 \), so by Poincaré duality \( H^2(M) = 0 \), so by the universal coefficient theorem \( H_2(M) = 0 \). So, by the Hurewicz theorem one more time, \( G = \pi_2(M) = 0 \). Now, if \( M \) is a \( K(\{0\}, 2) \) then \( \pi_i(M) = 0 \) for all \( i \) so by the Hurewicz theorem \( H_i(M) = 0 \) for all \( i \). In particular, \( H_3(M) = 0 \), which contradicts the fact that \( M \) was closed and orientable.

(9) Recall that orientable \( k \)-dimensional vector bundles over \( X \) are in bijection with \( [X, Gr_3^+(\mathbb{R}^\infty)] \), where \( Gr_3^+(\mathbb{R}^\infty) = V_3(\mathbb{R}^\infty)/SO(3) \) is the Grassmanian of oriented 3-planes in \( \mathbb{R}^\infty \). Compute \( \pi_i(Gr_3^+(\mathbb{R}^\infty)) \) for \( i \leq 4 \). (Hint: recall that \( SO(3) \cong \mathbb{R}P^3 \).)

**Solution.** The space \( V_3(\mathbb{R}^\infty) \) is contractible, so the long exact sequence for the fibration \( SO(3) \to V_3(\mathbb{R}^\infty) \to Gr_3^+(\mathbb{R}^\infty) \) decomposes as

\[
0 = \pi_n(V_3(\mathbb{R}^\infty) \to \pi_n(Gr_3^+(\mathbb{R}^\infty)) \to \pi_{n-1}(SO(3)) \to \pi_{n-1}(V_3(\mathbb{R}^\infty))) = 0.
\]

Hence, \( \pi_n(Gr_3^+(\mathbb{R}^\infty)) \cong \pi_{n-1}(SO(3)) \).

As noted in the hint, \( SO(3) \cong \mathbb{R}P^3 \). Hence, \( \pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z} \) and for \( i > 1 \), \( \pi_i(SO(3)) \cong \pi_i(S^3) \) (since \( S^3 \) is a covering space of \( \mathbb{R}P^3 \)). From the Hurewicz theorem, \( \pi_2(S^3) = 0 \) and \( \pi_3(S^3) \cong H_3(S^3) \cong \mathbb{Z} \). Hence, the first few homotopy groups of \( Gr_3^+(\mathbb{R}^\infty) \) are:

\[
\pi_i(Gr_3^+(\mathbb{R}^\infty)) = \begin{cases} 
0 & i = 0 \\
0 & i = 1 \\
\mathbb{Z}/2\mathbb{Z} & i = 2 \\
0 & i = 3 \\
\mathbb{Z} & i = 4.
\end{cases}
\]

(10) Let \( Y \) be a 2-connected space and \( p: Y \to Gr_3^+(\mathbb{R}^\infty) \) a fibration so that \( p_\ast: \pi_i(Y) \to \pi_i(Gr_3^+(\mathbb{R}^\infty)) \) is an isomorphism for \( i > 2 \). (That is, \( Y \) is a 2-connected cover of \( Gr_3^+(\mathbb{R}^\infty) \).) Define the (primary) obstruction in cohomology to lifting a map \( f: X \to \)}
\( \text{Gr}^+_{3}(\mathbb{R}^\infty) \) to a map \( \widetilde{f}: X \to Y \) and give an example where the obstruction does not vanish.

**Solution.** From the long exact sequence in homotopy groups, the fibration \( Y \to \text{Gr}^+_{3}(\mathbb{R}^\infty) \) has fiber \( K(\mathbb{Z}/2\mathbb{Z}, 1) \). Since \( \pi_1(Y) \cong \pi_1(\text{Gr}^+_{3}(\mathbb{R}^\infty)) = 0 \), the map \( Y \to \text{Gr}^+_{3}(\mathbb{R}^\infty) \) is a principal fibration. (This is a special case of the statement about Moore-Postnikov fibrations on the “possibly useful theorems” page, and is also immediate from the construction above.) So, a map \( f: X \to \text{Gr}^+_{3}(\mathbb{R}^\infty) \) has a lift if and only if the composite

\[
X \xrightarrow{f} \text{Gr}^+_{3}(\mathbb{R}^\infty) \xrightarrow{g} K(\mathbb{Z}/2\mathbb{Z}, 2)
\]

is nullhomotopic. The homotopy class \([g \circ f] \in [X, K(\mathbb{Z}/2\mathbb{Z}, 2)]\) is trivial if and only if

\[(g \circ f)^* (\iota) \in H^2(X; \mathbb{Z}/2\mathbb{Z})\]

vanishes. The element \((g \circ f)^* (\iota)\) is the primary obstruction to lifting \( f \).

For an example where the primary obstruction does not vanish, take \( X = \text{Gr}^+_{3}(\mathbb{R}^\infty) \) and let \( f \) be the identity map. Then from the construction in the previous solution \((g \circ \mathbb{1})^* (\iota) = g^* (\iota)\) is a generator of \( H^2(\text{Gr}^+_{3}(\mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z})\).