Final

Probability Qualifying Exam 2019

There are eight problems on this test. Read each problem carefully before beginning. PARTIAL CREDIT CANNOT BE AWARDED UNLESS YOUR WORK IS CLEAR.

Problem	Possible Points	Earned Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

Problem 1. Construct a sequence of random variables $\{X_n\}$ and X so that $X_n \to X$ in probability, but X_n does not converge to X almost surely.

Solution. Let $\Omega = \{0,1\}^\infty$ with the product σ-algebra. Set $\mathbb{P}$ be the product measure with n-th marginal giving mass $1/n$ to 1, and let X_n be the n-th coordinate mapping.

Then $\mathbb{P}(X_n \neq 0) = 1/n \to 0$, so $\{X_n\}$ converges in probability to 0. On the other hand, the second Borel-Cantelli Lemma implies that $\mathbb{P}(X_n = 1 \text{ i.o.}) = 1$, as $\sum_n 1/n = \infty$, whence $\{X_n\}$ does not converge to 0 almost surely. $\square$

Problem 2. For $\{X_k\}$ a sequence of random variables with $\mathbb{E}(|X_k|) < B$ and $\mathbb{E}(X_k) = \mu$ for all k, let $\mathscr{F}_n = \sigma(X_1, \dots, X_n)$, and $S_n = \sum_{k=1}^n X_k$. Suppose that X_k is independent of $\mathscr{F}_{k-1}$. Suppose that T a is $\{1, 2, \dots\}$-valued stopping time for $\{\mathscr{F}_n\}$ with $\mathbb{E}(T) < \infty$. Show that

$$\mathbb{E}(S_T) = \mathbb{E}(T)\mu .$$

Hint: $S_T = \sum_k X_k \mathbb{1}\{T \geq k\}$

Solution. First, we have

$$\mathbb{E}\left(\sum_k |X_k| \mathbb{1}\{T \geq k\}\right) = \sum_k \mathbb{E}(|X_k| \mathbb{1}\{T \geq k\}) .$$

Since $\{T \geq k\} = \{T \leq k-1\}^c \in \mathscr{F}_{k-1}$, by independence of $|X_k|$ and $\mathscr{F}_{k-1}$, it follows that the right-hand side above equals

$$\sum_k \mathbb{P}\{T \geq k\} \mathbb{E}(|X_k|) \leq B\mathbb{E}(T) .$$

We can therefore apply Fubini to obtain

$$\mathbb{E}\left(\sum_k X_k \mathbb{1}\{T \geq k\}\right) = \sum_k \mathbb{E}(X_k) \mathbb{P}(T \geq k) = \mu \mathbb{E}(T) .$$

□

Problem 3. Let $\{X_k\}$ be an i.i.d. sequence of $\{0,1\}$ random bits, i.e. random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = 0) = 1/2$.

Let

$$L_n = \max\{m \geq n : X_n = X_{n+1} = \cdots = X_m = 1\}$$

be the run of $+1$'s beginning at n, and let

$$R_n = \max_{k \leq n} L_k$$

be longest run beginning before or at n, respectively.

(i) Show that if $p > 1$ and $\varepsilon > 1/p$, then

$$\limsup_{n\to\infty} \frac{R_{n^p}}{\log_2 n^p} \leq 1 + \varepsilon\,.$$

(ii) Use (i) to show that

$$\limsup_{n\to\infty} \frac{R_n}{\log_2 n} \leq 1\,.$$

(iii) By considering disjoing blocks of bits $(X_k, X_{k+1}, \ldots, X_{k+r-1})$ of length $r = (1-\varepsilon)\log_2 n$, show that

$$\liminf_{n\to\infty} \frac{R_n}{\log_2 n} \geq 1\,.$$

Solution. First, for any $\varepsilon > 0$, subadditivity yields

$$\mathbb{P}(R_n > (1+\varepsilon)\log_2 n) = \mathbb{P}\left(\bigcup_{k\leq n}\{L_k > (1+\varepsilon)\log_2 n\}\right) \leq n\mathbb{P}(L_1 > (1+\varepsilon)\log_2 n) = n\left(\frac{1}{2}\right)^{(1+\varepsilon)\log_2 n} = n^{-\varepsilon}\,.$$

Thus substituting n^p for n in the above,

$$\mathbb{P}(R_{n^p} > (1+\varepsilon)\log_2 n^p) \leq n^{-\varepsilon p}\,,$$

and if $np > 1$, then Borel-Cantelli implies that

$$\limsup_{n} \frac{R_{n^p}}{\log_2 n^p} \leq (1+\varepsilon)\,.$$

For $n^p < k \le (n+1)^p$, since R_n is non-decreasing,

$$\frac{R_k}{\log_2 k} \le \frac{R_{(n+1)^p}}{\log_2 n^p} = \frac{\log_2(n+1)}{\log_2 n} \frac{R_{(n+1)^p}}{\log_2(n+1)^p},$$

and taking limits superior, since $\frac{\log(n+1)}{\log n} \to 1$,

$$\limsup \frac{R_n}{\log_2 n} \le 1 + \varepsilon.$$

Taking $\varepsilon \downarrow 0$ finishes the proof.

On the other hand,

$$\begin{aligned}\mathbb{P}(R_n < (1-\varepsilon)\log_2 n) &\le \mathbb{P}(\text{each disjoint block of size } (1-\varepsilon)\log_2 n \text{ contains a } 0) \\ &= [1 - 2^{-(1-\varepsilon)\log_2 n}]^{n/(1-\varepsilon)\log_2 n} \\ &\le e^{-n^{\varepsilon}/(1-\varepsilon)\log_2 n},\end{aligned}$$

and so by Borel-Cantelli, we have that $\mathbb{P}(R_n < (1-\varepsilon)\log_2 n \text{ i.o.}) = 0$. In other words,

$$\liminf_{n\to\infty} \frac{R_n}{\log_2 n} > 1 - \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we have

$$\liminf_{n\to\infty} \frac{R_n}{\log_2 n} \ge 1.$$

□

Problem 4. Suppose that $\{X_n\}$ is an irreducible Markov chain on a countable set S. Suppose that for some x_0 and a non-negative function $f: S \to (0,\infty)$ there is a constant $0 < \alpha < 1$ satisfying

$$\mathbb{E}_x[f(X_1)] \le \alpha f(x) \quad \text{for all } x \neq x_0 .$$

Suppose further that $f(x_0) \le f(x)$ for all x. Show that $\{X_n\}$ is positive recurrent.

Solution. If $M_n = f(X_n)/\alpha^n$, then $\{M_{n\wedge\tau}\}$ is a non-negative supermartingale, where

$$\tau = \inf\{n \ge 0 : X_n = x_0\} .$$

We have by the supermartingale property,

$$f(x_0)\alpha^{-n}\mathbb{P}(\tau > n) \le \mathbb{E}_x(M_n \mathbb{1}\{\tau > n\}) \le \mathbb{E}_x(M_{n\wedge\tau}) \le f(x) .$$

That is,

$$\mathbb{P}(\tau > n) \le [f(x)/f(x_0)]\alpha^n .$$

We conclude that $\mathbb{E}_x(\tau) = \sum_n \mathbb{P}(\tau > n) < \infty$. By irreducibility, the chain is positive recurrent.

□

Problem 5. Prove the following directly, i.e., without referencing any version of the Central Limit Theorem: Let $\{X_k\}$ be i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^2) = 1$. Prove that $S_n/\sqrt{n}$ converges in law to a standard Normal distribution.

Proof. We can use the identity

$$e^{ix} = 1 + ix - \frac{x^2}{2} + R(x),$$

where

$$|R(x)| \le |x|^2 \wedge |x|^3 = |x|^2(1 \wedge |x|).$$

First, letting $\phi(t) = \mathbb{E}(e^{itX_1})$,

$$\begin{aligned}\phi(t) &= \mathbb{E}\left(1 + itX_1 - \frac{t^2}{2}X_1^2 + R(tX_1)\right) \\ &= 1 + it\mathbb{E}(X_1) - \frac{t^2}{2}\mathbb{E}(X_1^2) + \mathbb{E}R(tX_1) \\ &= 1 - \frac{t^2}{2} + \mathbb{E}R(tX_1).\end{aligned}$$

Let $r(t) = \mathbb{E}[R(tX_1)]$.

Let ϕ_n be the ch.f. of $S_n/\sqrt{n}$:

$$\phi_n(t) = \mathbb{E}\exp\left(\frac{\sum_{k=1}^n X_k}{\sqrt{n}}t\right) = \prod_{k=1}^n \mathbb{E}\exp\left(X_k\frac{t}{\sqrt{n}}\right) = \phi\left(\frac{t}{\sqrt{n}}\right)^n.$$

Note that

$$nr(t/\sqrt{n}) = t^2\mathbb{E}[X_1^2(1 \wedge |X_1 t|/\sqrt{n}))] \to 0,$$

by the Lebesgue Dominated Convergence Theorem.

We next use the inequality $|z_1 z_2 \cdots z_n - w_1 \cdots w_n| \le \sum |z_i - w_i|$, valid for $|z_i| \le 1$ and $|w_i| \le 1$:

$$|\phi(t/\sqrt{n})^n - (1 - t^2/n)^n| \le nr(t/\sqrt{n}) \to 0.$$

Since $(1 - t^2/n)^n \to e^{-t^2/2}$, we have that

$$\phi(t/\sqrt{n})^n \to e^{-t^2/2},$$

which by the Continuity Theorem, implies that $S_n/\sqrt{n}$ converges to the standard Normal law. □

Problem 6. Let $\{B_t\}$ be a Brownian motion in $\mathbb{R}^2$, started at x with $r < |x| < R$. Let τ be the first hitting time on the boundary of the annulus of radii r and R.

(i) Find $\mathbb{P}_x(B_\tau \in S_r)$, where S_r is the circle of radius r.

Hint: Consider the process $M_t = \log|B_t|$

(ii) Show that two-dimensional Brownian motion does not hit points.

Solution. Let $f(x) = \log|x|$. We know that $\Delta f = 0$, so then by Itô's Lemma, $M_t = \log|B_t|$ is a local martingale. Because $M_{t\wedge\tau}$ is bounded, we know that $M_{t\wedge\tau}$ is a true martingale.

Thus, since $\tau < \infty$ (by, e.g., recurrence of one-dimensional Brownian motion), we can apply optional stopping theorem, and then the Bounded Convergence Theorem to obtain

$$\log|x| = \mathbb{E}_x \log|B_{t\wedge\tau}| \overset{t\to\infty}{\longrightarrow} \mathbb{E}_x \log|B_\tau| = \log r\,\alpha + \log R(1-\alpha)\,,$$

where $\alpha = \mathbb{P}_x(B_\tau \in S_r)$.

Then

$$\alpha = \frac{\log R - \log|x|}{\log R - \log r}\,.$$

As $r \downarrow 0$, this goes to 0, whence the brownian motion does not hit 0. Translation invariance gives that brownian motion does not hit any point (other than its starting position).

□

Problem 7. Let $\{X_n\}$ be the nearest neighbor walk on $\{0,1,\ldots\}$ with $P(k,k-1) = q > P(k,k+1) = p$ for $k > 0$. Also, $P(0,0) = q$. Find $\mathbb{E}_0\tau_0$.

Solution. The stationary distribution is geometric: $\pi(k) = (1-\alpha)\alpha^k$, where $\alpha = (p/q)$:

$$\alpha^{k-1}p + \alpha^{k+1}q = \alpha^k(p(q/p) + q(p/q)) = \alpha^k$$

for $k \geq 1$, and

$$q + q(p/q) = 1\,,$$

verifying the stationary equation at 0.

Thus, by the Kac identity, $\mathbb{E}_0\tau_0 = \pi(0) = 1/(1-\alpha)$. □

Problem 8. Let $\{B_t\}$ be a standard Brownian motion. Show that $\sup_s B_s = \infty$.

Proof. Let $M = \sup_s B_s$. Let $\tilde{B}_t = cB_{s/c^2}$. This is also a standard Brownian motion. Thus M and $\tilde{M}$ have the same distribution. On the other hand, $\tilde{M} = cM$. It follows that M is supported on $\{0, \infty\}$.

Note that by the Markov property,

$$\mathbb{P}(\sup_t B_{t+1} - B_1 \leq -B_1 \mid \mathscr{F}_1) = \phi(B_1)\,,$$

where

$$\phi(x) = \mathbb{P}(M \leq -x)\,.$$

Since M is supported on $\{0, \infty\}$, we have $\phi(x) = \mathbb{P}(M = 0)$ for $x < 0$.

Thus,

$$\mathbb{P}(M = 0) \leq \mathbb{P}(B_1 \leq 0,\ \sup_t B_{t+1} - B_1 \leq -B_t) = \mathbb{P}(B_1 \leq 0)\mathbb{P}(M = 0) = \frac{1}{2}\mathbb{P}(M = 0)\,.$$

Thus $\mathbb{P}(M = 0) = 0$, and $M = \infty$ a.s.

(Alternatively, one can use or prove that $B_t > 0$ for some $t \in (0, \varepsilon)$ for all ε.)

□