Probability Qualifying Exam 2019

There are eight problems on this test. Read each problem carefully before beginning. PARTIAL CREDIT CANNOT BE AWARDED UNLESS YOUR WORK IS CLEAR.

Problem	Possible Points	Earned Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

Problem 1. Construct a sequence of random variables $\{X_n\}$ and X so that $X_n \to X$ in probability, but X_n does not converge to X almost surely.

Problem 2. For $\{X_k\}$ a sequence of random variables with $\mathbb{E}(|X_k|) < B$ and $\mathbb{E}(X_k) = \mu$ for all k, let $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$, and $S_n = \sum_{k=1}^n X_k$. Suppose that X_k is independent of $\mathscr{F}_{k-1}$. Suppose that T a is $\{1, 2, \ldots\}$-valued stopping time for $\{\mathscr{F}_n\}$ with $\mathbb{E}(T) < \infty$. Show that

$$\mathbb{E}(S_T) = \mathbb{E}(T)\mu.$$

Hint: $S_T = \sum_k X_k \mathbb{1}\{T \geq k\}$

Problem 3. Let $\{X_k\}$ be an i.i.d. sequence of $\{0,1\}$ random bits, i.e. random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = 0) = 1/2$.

Let

$$L_n = \max\{m \geq n : X_n = X_{n+1} = \cdots = X_m = 1\}$$

be the run of $+1$'s beginning at n, and let

$$R_n = \max_{k \leq n} L_k$$

be longest run beginning before or at n, respectively.

(i) Show that if $p > 1$ and $\varepsilon > 1/p$, then, a.s.,

$$\limsup_{n\to\infty} \frac{R_{n^p}}{\log_2 n^p} \leq 1 + \varepsilon \,.$$

(ii) Use (i) to show that, a.s.,

$$\limsup_{n\to\infty} \frac{R_n}{\log_2 n} \leq 1 \,.$$

(iii) By considering disjoing blocks of bits $(X_k, X_{k+1}, \ldots, X_{k+r-1})$ of length $r = (1-\varepsilon)\log_2 n$, show that, a.s.,

$$\liminf_{n\to\infty} \frac{R_n}{\log_2 n} \geq 1 \,.$$

Problem 4. Suppose that $\{X_n\}$ is an irreducible Markov chain on a countable set S. Suppose that for some x_0 and a non-negative function $f : S \to (0,\infty)$ there is a constant $0 < \alpha < 1$ satisfying

$$\mathbb{E}_x[f(X_1)] \leq \alpha f(x) \quad \text{for all } x \neq x_0 \,.$$

Suppose further that $f(x_0) \leq f(x)$ for all x. Show that $\{X_n\}$ is positive recurrent.

Problem 5. Prove the following directly, i.e., without referencing any version of the Central Limit Theorem: Let $\{X_k\}$ be i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^2) = 1$. Prove that $S_n/\sqrt{n}$ converges in law to a standard Normal distribution.

Problem 6. Let $\{B_t\}$ be a Brownian motion in $\mathbb{R}^2$, started at x with $r < |x| < R$. Let τ be the first hitting time on the boundary of the annulus of radii r and R.

(i) Find $\mathbb{P}_x(B_\tau \in S_r)$, where S_r is the circle of radius r.

Hint: Consider the process $M_t = \log|B_t|$

(ii) Show that two-dimensional Brownian motion does not hit points, almost surely.

Problem 7. Let $\{X_n\}$ be the nearest neighbor walk on $\{0,1,\ldots\}$ with $P(k,k-1) = q > P(k,k+1) = p$ for $k > 0$. Also, $P(0,0) = q$. Find $\mathbb{E}_0\tau_0$.

Problem 8. Let $\{B_t\}$ be a standard Brownian motion. Show that $\sup_s B_s = \infty$, almost surely.