1. Differential Geometry quals Problems for Winter 2019

Problem 1.1. Let ω be a smooth 1-form and let X and Y be smooth vector fields. Prove that $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$.

Solution. Fix ω. Let $\mathcal{T}(X,Y) = \{d\omega\}(X,Y) - \Big(X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\Big)$. We note that ω and $d\omega$ are tensors.

$$\begin{aligned} & \mathcal{T}(fX,Y) = \{d\omega\}(X,Y) - \Big(X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\Big) \\ = \; & f\{d\omega\}(X,Y) - fX(\omega(Y)) + Y(f\omega(X)) + \omega(f[X,Y] - Y(f)X) \\ = \; & f\{d\omega\}(X,Y) - fX(\omega(Y)) + Y(f\omega(X)) + f\omega([X,Y]) - Y(f)\omega(X) \\ = \; & f\mathcal{T}(X,Y)\,. \end{aligned}$$

Since $\mathcal{T}(X,Y) = -\mathcal{T}(Y,X)$, we also have $\mathcal{T}(X,fY) = f\mathcal{T}(X,Y)$. Thus $\mathcal{T}$ is a tensor. Thus to show $\mathcal{T}$ vanishes, it suffices to check it on a coordinate frame ∂_{x^i}. Let $\omega = a_i dx^i$. We compute

$$\begin{aligned} & (d\omega)(\partial_{x^k}, \partial_{x^\ell}) = (\partial_{x^j} a_i dx^j \wedge dx^i)(\partial_{x^k}, \partial_{x^\ell}) = \partial_{x^j}(a_i)\{\delta^j_k\delta^\ell_i - \delta^j_l\delta^i_k\} = \partial_{x^k} a_\ell - \partial_{x^\ell} a_k, \\ & X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = \partial_{x^k}(a_l) - \partial_{x^l}(a_k) - 0\,. \end{aligned}$$

Problem 1.2. Let T be an isometry of a compact Riemannian manifold (M,g). Let S be a connected component of the fixed point set of T. Show S is a smooth submanifold of M which is totally geodesic.

Solution. Let $P \in S$. Let $\mathcal{T} := \{\xi \in T_P M : T_*\xi = \xi\}$. This is a linear subspace of $T_P M$. There exists $\epsilon > 0$ so that $\exp_P : \{\xi \in T_P M : \|\xi\| < \epsilon\} \to M$ is a diffeomorphism onto its image $B_\epsilon(P)$. ϵ can also be chosen so that if $d(P,Q) < \epsilon$, then there is a unique geodesic from P to Q lying in $B_\epsilon(P)$. Thus $\exp_P(\mathcal{T} \cap B_\epsilon(0))$ is a smooth diffeomorphism to its range $\tilde{S}$. If $\xi \in \mathcal{T}$, then the curve $\exp_P(t\xi)$ is a geodesic in M. Since the initial point and initial direction of this geodesic is fixed by T, the whole geodesic is fixed by T and hence lies in S. Thus $\tilde{S} \subset S$. On the other hand, if $Q \in B_\epsilon(P)$ and $TQ = Q$, then there is a unique geodesic from P to Q and hence this geodesic must be fixed by T. Thus the initial direction of this geodesic must be fixed and hence the initial direction lies in $\mathcal{T}$. Thus Q is in $\tilde{S}$. Thus $S = \tilde{S}$ near P and S is totally geodesic. □

Problem 1.3. Let $\mathcal{O}$ be a connected open subset of $\mathbb{R}^2$. Let f be a smooth function on $\mathcal{O}$ and let $ds^2 = e^{2f}(dx^2 + dy^2)$. Determine the scalar curvature τ. Use the resulting formula to find the Gaussian curvature of the upper half plane with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$.

Solution. Suppose $ds^2 = e^{2f}(dx^2 + dy^2)$. We use the Koszul formula to express

$$\begin{aligned} & g(\nabla_{\partial_x}\partial_x, \partial_x) = f_x e^{2f}, && g(\nabla_{\partial_x}\partial_x, \partial_y) = -f_y e^f, && \nabla_{\partial_x}\partial_x = f_x\partial_x - f_y\partial_y, \\ & g(\nabla_{\partial_x}\partial_y, \partial_x) = f_y e^{2f}, && g(\nabla_{\partial_x}\partial_y, \partial_y) = f_x e^f, && \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = f_y\partial_x + f_x\partial_y, \\ & g(\nabla_{\partial_y}\partial_y, \partial_x) = -f_x e^{2f}, && g(\nabla_{\partial_y}\partial_y, \partial_y) = f_y e^f, && \nabla_{\partial_y}\partial_y = -f_x\partial_x + f_y\partial_y. \end{aligned}$$

We compute

$$\begin{aligned} & (\nabla_{\partial_x}\nabla_{\partial_y} - \nabla_{\partial_y}\nabla_{\partial_x} - \nabla_{[\partial_x,\partial_y]})\partial_y \\ = \; & \nabla_{\partial_x}\{-f_x\partial_x + f_y\partial_y\} - \nabla_{\partial_y}\{f_y\partial_x + f_x\partial_y\} \\ = \; & -f_{xx}\partial_x - f_x(f_x\partial_x - f_y\partial_y) + f_{xy}\partial_y + f_y\{f_y\partial_x + f_x\partial_y\} \\ & -f_{yy}\partial_x - f_y\{f_y\partial_x + f_x\partial_y\} - f_{xy}\partial_y - f_x\{-f_x\partial_x + f_y\partial_y\} \\ = \; & (-f_{xx} - f_{yy} - f_x f_x + f_y f_y - f_y f_y + f_x f_x)\partial_x \\ & +(f_x f_y + f_{xy} + f_y f_x - f_y f_x - f_{xy} - f_x f_y)\partial_y \\ = \; & -(f_{xx} + f_{yy})\partial_x\,. \end{aligned}$$

We then have

$$\tau = 2\frac{g(R(\partial_x,\partial_y)\partial_y,\partial_x)}{g(\partial_x,\partial_x)g(\partial_y,\partial_y)-g(\partial_x,\partial_y)^2} = -2e^{-2f}(f_{xx}+f_{yy})\,.$$

Suppose $ds^2 = \frac{dx^2+dy^2}{y^2}$. Then $e^{2f} = y^{-2}$ so $2f = -2\log(y)$ and $f = -\log(y)$. We have $f_{xx}+f_{yy} = y^{-2}$ and $-2e^{-2f}(f_{xx}+f_{yy}) = -2y^2(y^{-2}) = -2$. That is correct.

Problem 1.4. Let $\mathcal{M}$ be a Riemannian manifold.

(1) Let $\{e_1,\ldots,e_m\}$ be a local orthonormal frame for the tangent bundle. Let ∇ be the Levi-Civita connection. Determine constants $\{a,b,c\}$ so that $g(\nabla_{e_i}e_j,e_k) = a*g([e_i,e_j],e_k)+b*g([e_j,e_k],e_i)+c*g([e_k,e_i],e_j)$.

(2) Show that there exists a local orthonormal frame field with $[e_i,e_j]=0$ for all i,j if and only if the curvature tensor vanishes identically.

Solution. We compute:

$$\begin{aligned}
& g(\nabla_{e_i}e_j,e_k) = g([e_i,e_j],e_k)+g(\nabla_{e_j}e_i,e_k) \text{ (torsion free)}\\
= \;& g([e_i,e_j],e_k)+e_jg(e_i,e_k)-g(e_i,\nabla_{e_j}e_k) \text{ (Riemannian)}\\
= \;& g([e_i,e_j],e_k)-g(e_i,[e_j,e_k])-g(e_i,\nabla_{e_k}e_j) \text{ (torsion free)}\\
= \;& g([e_i,e_j],e_k)-g(e_i,[e_j,e_k])-e_kg(e_i,e_j)+g(\nabla_{e_k}e_i,e_j) \text{ (Riemannian)}\\
= \;& g([e_i,e_j],e_k)-g(e_i,[e_j,e_k])+g([e_k,e_i],e_j)+g(\nabla_{e_i}e_k,e_j) \text{ (torsion free)}\\
= \;& g([e_i,e_j],e_k)-g(e_i,[e_j,e_k])+g([e_k,e_i],e_j)-e_ig(e_k,e_j)-g(e_k,\nabla_{e_i}e_j) \text{ (Riemannian)},\\
& 2g(\nabla_{e_i}e_j,e_k) = g([e_i,e_j],e_k)-g(e_i,[e_j,e_k])+g([e_k,e_i],e_j).
\end{aligned}$$

So $a=\frac{1}{2}$, $b=-\frac{1}{2}$, and $c=\frac{1}{2}$.

If the curvature tensor vanishes identically, then M has constant sectional curvature 0. So M is locally isometric to flat space. The coordinates frame satisfies $g(\partial_{x^i},\partial_{x^j}) = \delta_{ij}$ and $[\partial_{x^i},\partial_{x^j}]=0$. This provides the requisite torsion free orthonormal basis. Conversly, if such an orthonormal basis exists, then we have that $g(\nabla_{e_i}e_j,e_k)=0$ for all i,j,k. This implies $\nabla_{e_i}e_j=0$ for all i,j. Expand $[e_i,e_j] = C_{ij}{}^{\ell}e_\ell$. Then

$$R(e_i,e_j)e_k = \nabla_{e_i}\nabla_{e_j}e_k-\nabla_{e_j}\nabla_{e_i}e_k-\nabla_{[e_i,e_j]}e_k = \nabla_{e_i}0-\nabla_{e_j}0-C_{ij}{}^{\ell}\nabla_{e_\ell}e_k = 0\,.$$

Problem 1.5. Let $\mathbb{Z}_\ell = \{\lambda\in\mathbb{C}:\lambda^\ell=1\}$ be the cyclic group of ℓ^{th} roots of unity. Let $S^{2n-1} := \{\vec{z}=(z^1,\ldots,z^n)\in\mathbb{C}^n : |\vec{z}|^2=1\}$ be the unit sphere. Let $\mathbb{Z}_\ell$ act on S^{2n-1} by complex multiplication. The quotient $M_{n,\ell} := S^{2n-1}/\mathbb{Z}_\ell$ is called a classical or diagonal lens space. It is a smooth manifold - you need not prove this. Determine the de Rham cohomology of $M_{n,\ell}$.

Solution. Let Γ be a finite group acting by smooth isometries and without fixed points on a compact manifold M. We showed that $H^p_{\text{DeR}}(M/\Gamma) = H^p_{\text{DeR}}(M)^\Gamma$ is the space of invariant cohomology classes. Let $\lambda(t) = e^{2\pi\sqrt{-1}t}$ act on S^{2n-1} by complex multiplication to define a smooth 1-parameter family of diffeomorphisms of S^{2n-1}. Since $\lambda(0)$ is the the identity, $\lambda(t)$ is the identity acting on $H^p_{\text{DeR}}(S^{2n-1})$ for any P. Thus $H^p_{\text{DeR}}(S^{2n-1}/\mathbb{Z}_\ell)$ is isomorphic to $H^p_{\text{DeR}}(S^{2n-1})$ for any p. This is $\mathbb{R}$ if $p=0$ or if $p=2n-1$; it is zero otherwise.

Problem 1.6. A 3-dimensional connected Lie group G has a Lie algebra $\mathfrak{g}$ where relative to a suitable basis we have $[e_1,e_3]=e_1$ and $[e_2,e_3]=-e_2$. Determine the associated Lie algebra cohomology. Can G be compact? Justify your answer.

Solution: We use the formula $d\omega(e_a\wedge e_b) = -\omega([e_a,e_b])$ to compute:

$$\begin{aligned}
& d\mathbb{1}=0, \quad de^1 = -e^1\wedge e^3, && de^2 = e^2\wedge e^3, \quad de^3=0,\\
& d(e^1\wedge e^2) = -e^1\wedge e^3\wedge e^2 - e^1\wedge e^2\wedge e^3 = 0, && d(e^2\wedge e^3)=0, \quad d(e^1\wedge e^3)=0,\\
& d(e^1\wedge e^2\wedge e^3)=0.
\end{aligned}$$

Consequently $H^*(\mathfrak{g}) = \mathbb{R}\cdot\mathbb{1}\oplus\mathbb{R}\cdot e^3\oplus\mathbb{R}\cdot e^1\wedge e^2\oplus\mathbb{R}\cdot e^1\wedge e^2\wedge e^3$. If there exists a compact group G whose Lie algebra is $\mathfrak{g}$, then $H^*(G) = H^*(\mathfrak{g})$. Furthermore, $H^*(G)$ is an exterior algebra on odd dimensional generators; in particular $H^2(G)$ is generated by $H^1(G)$. This is not the case for $H^*(\mathfrak{g})$ and hence no such compact group exists.

Problem 1.7. Show that any complex manifold is orientable.

Solution. The following probably is overkill and I would accept a solution which simply assumed the result from linear algebra. We begin with a result from linear algebra. We may identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. This identification defines an embedding $\iota : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$. If $T \in \mathrm{GL}(n,\mathbb{C})$, show that $|\det(T)|^2 = \det(\iota(T))$. Suppose first $n = 1$. Let e_1 be the standard basis for $\mathbb{C}$ so $\{e_1, \sqrt{-1}e_1\}$ is the corresponding basis for $\mathbb{R}^2$. Then

$$T(e_1) = a_1e_1 + \sqrt{-1}a_2e_1 \text{ and } T(\sqrt{-1}e_1) = \sqrt{-1}T(e_1) = -a_2e_1 + \sqrt{-1}a_2e_1\,.$$

Thus $|\det(T)|^2 = a_1^2 + a_2^2$ and

$$\det(\iota(T)) = \det\begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1\end{pmatrix} = a_1^2 + a_2^2\,.$$

Suppose next n is arbitrary and $T = \mathrm{diag}(\alpha_1,\ldots,\alpha_n)$ is diagonal. Then we establish the desired identity by computing

$$\begin{aligned} |\det(T)|^2 &= |\det(\mathrm{diag}(\alpha_1\ldots\alpha_n))|^2 = |\alpha_1\ldots\alpha_n|^2 = |\alpha_1^2|\ldots|\alpha_n^2|,\\ \det(\iota(T)) &= \det(\iota(\alpha_1\,\mathrm{id})\oplus\cdots\oplus\iota(\alpha_n\,\mathrm{id}))\\ &= \det(\iota(\alpha_1\,\mathrm{id}))\ldots\det(\iota(\alpha_n\,\mathrm{id}) = |\alpha_1|^2\ldots|\alpha_n|^2\,. \end{aligned}$$

Suppose next n is arbitrary and T is diagonalizable so there exists $g\in\mathrm{GL}(n,\mathbb{C})$ so that $T = g\,\mathrm{diag}(\alpha_1,\ldots,\alpha_n)g^{-1}$. We compute

$$\begin{aligned} |\det(T)|^2 &= |\det(gTg^{-1})|^2 = |\alpha_1\ldots\alpha_n|^2 = |\alpha_1^2|\ldots|\alpha_n^2|\\ &= \det(\iota(gTg^{-1}) = \det(\iota(g)\iota(T)\iota(g)^{-1}) = \det(\iota(T))\,. \end{aligned}$$

This is where I would expect the average student to start If M is holomorphic, the transition functions $\Phi_{\alpha\beta}$ are holomorphic and thus $d\Phi_{\alpha\beta}\in\mathrm{GL}(n,\mathbb{C})$. The corresponding real transition functions are then given by $\iota\{d\Phi_{\alpha\beta}\}$. Thus we have $\det\{\iota(d\Phi_{\alpha\beta}\} = |\det(d\Phi_{\alpha\beta})|^2 > 0$ and M is orientable.

Problem 1.8. Let $\mathbb{RP}^m = S^m/\mathbb{Z}_2$ be real projective space. Let $\mathbb{L}$ be the real line bundle over $\mathbb{RP}^m$ defined by taking $(S^m\times\mathbb{R})/\mathbb{Z}_2$ where we identify $(\vec{x},\lambda)$ with $(-\vec{x},-\lambda)$ for $\vec{x}\in S^m$ and $\lambda\in\mathbb{R}$. Let $\mathbb{1} = \mathbb{RP}^m\times\mathbb{R}$ be the trivial real line bundle over $\mathbb{RP}^m$ and let $T(\mathbb{RP}^m)$ be the tangent bundle of $\mathbb{RP}^m$. Construct an isomorphism between the stable tangent bundle $T(\mathbb{RP}^m)\oplus\mathbb{1}$ and the direct sum of $(m+1)$ copies of $\mathbb{L}$.

Solution. Let ν be the unit normal to the sphere; $\nu(\vec{x}) = \vec{x}$ for $\vec{x}\in S^m\subset\mathbb{R}^{m+1}$. Let $\xi\in T(S^m)$. Then $T(S^m) = \{(\vec{x},\vec{t})\in S^m\times\mathbb{R}^{m+1} : \vec{x}\perp\vec{t}\}$ and

$$\begin{aligned} T(S^m)\oplus\mathbb{1}_{S^m} &= \{(\vec{x},\vec{t},\lambda)\} : \vec{x}\perp\vec{t}\text{ and }\lambda\in\mathbb{R}\},\\ \mathbb{1}_{S^m}^{m+1} &= \{(\vec{x},\vec{s}) : \vec{x}\in S^m \text{ and } \vec{s}\in\mathbb{R}^{m+1}\}\,. \end{aligned}$$

Define an isomorphism Φ from $T(S^m)\oplus\mathbb{1}_{S^m}$ to $\mathbb{1}_{S^m}^{m+1}$ by setting

$$\Phi(\vec{x},\vec{t},\lambda) = (\vec{x},\vec{t}+\lambda\vec{x})\,.$$

Note that $\Phi(-\vec{x},-\vec{t},\lambda) = -\Phi(\vec{x},\vec{t},\lambda)$. Thus Φ descends to an isomorphism between $T(\mathbb{RP}^m)\oplus 1_{\mathbb{RP}^m}$ and $\mathbb{L}^{m+1}$.

Problem 1.9. Let $T(\theta_1,\theta_2) = (\cos(\theta_1)(\cos(\theta_2)+2), \sin(\theta_1)(\cos(\theta_2)+2), \sin(\theta_2))$ for $0 \le \theta_1 \le 2\pi$ and $0 \le \theta_2 \le 2\pi$ parametrize a toroidal surface of revolution. Determine the first fundamental form I, the second fundamental form II, the principal curvatures, and the Gauss curvature.

Solution.

$$\begin{aligned}
&\partial_{\theta_1}T = (\cos(\theta_2)+2)(-\sin(\theta_1), \cos(\theta_1), 0),\\
&\partial_{\theta_2}T = (-\sin(\theta_2)\cos(\theta_1), -\sin(\theta_2)\sin(\theta_1), \cos(\theta_2)),\\
&g(\partial_{\theta_1},\partial_{\theta_1}) = (\cos(\theta_2)+2)^2, \quad g(\partial_{\theta_1},\partial_{\theta_2}) = 0, \quad g(\partial_{\theta_2},\partial_{\theta_2}) = 1,\\
&\partial_{\theta_1}T \times \partial_{\theta_2}T = (\cos(\theta_2)+2)(\cos(\theta_1)\cos(\theta_2), \sin(\theta_1)\cos(\theta_2), \sin(\theta_2)),\\
&N = (\cos(\theta_1)\cos(\theta_2), \sin(\theta_1)\cos(\theta_2), \sin(\theta_2)),\\
&\partial_{\theta_1}\partial_{\theta_1}T = (\cos(\theta_2)+2)(-\cos(\theta_1), -\sin(\theta_1), 0),\\
&\partial_{\theta_1}\partial_{\theta_2}T = -\sin(\theta_2)(-\sin(\theta_1), \cos(\theta_1), 0),\\
&\partial_{\theta_2}\partial_{\theta_2}T = (-\cos(\theta_2)\cos(\theta_1), -\cos(\theta_2)\sin(\theta_1), -\sin(\theta_2)),\\
&L_{11} = \partial_{\theta_1}\partial_{\theta_1}T \cdot N = -(\cos(\theta_2)+2)\cos(\theta_2),\\
&L_{12} = \partial_{\theta_1}\partial_{\theta_2}T \cdot N = 0,\\
&L_{22} = \partial_{\theta_2}\partial_{\theta_2}T \cdot N = -1,\\
&\kappa_1 = L_{11}/g_{11} = -\tfrac{\cos(\theta_2)}{2+\cos(\theta_2)}, \quad \kappa_2 = -1, \quad \kappa = \tfrac{\cos(\theta_2)}{\cos(\theta_2)+2}.
\end{aligned}$$

Problem 1.10. Use Stokes Theorem to prove the fundamental theorem of algebra.

Solution Let $p(z) = a_n z^n + \cdots + a_0$ be a complex polynomial where $n \ge 1$ and $a_n \ne 0$. Assume $p(z) \ne 0$ for all $z \in \mathbb{C}$ and argue at length for a contradiction. Without loss of generality, we may assume $a_n = 1$. If $\gamma : S^1 \to \mathbb{C} - \{0\}$, define

$$W(\gamma) := \frac{1}{2\pi}\int_\gamma \omega \text{ for } \omega := \frac{xdy - ydx}{x^2+y^2}\,.$$

Verify $d\omega = 0$. Let $\Gamma : S^1 \times [0,1] \to \mathbb{C} - \{0\}$. Let $\gamma_i(\theta) = \Gamma(\theta, i)$. Since $d\omega = 0$, we can use Stoke's theorem to see

$$\begin{aligned}
0 &= \int_{S^1\times[0,1]} \Gamma^* d\omega = \int_{S^1\times[0,1]} d\Gamma^*\omega\\
&= \int_{S^1\times\{1\}} \gamma_1^*\omega - \int_{S^1\times\{0\}} \gamma_0^*\omega = 2\pi\{W(\gamma_1) - W(\gamma_0)\}\,.
\end{aligned}$$

Let $\Gamma(\theta, r) = p(re^{\sqrt{-1}\theta})$. Then Γ_0 is the trivial path. Thus $W(\gamma_0) = 0$. Let $R = |a_{n-1}| + \cdots + |a_0| + 72$; $W(\Gamma_R) = W(\gamma_0) = 0$. For $\epsilon \in [0,1]$, define

$$\Theta(\theta,\epsilon) = \epsilon R^n e^{\sqrt{-1}n\theta} + (1-\epsilon)p(Re^{\sqrt{-1}\theta})\,.$$

We must show Θ takes values in $\mathbb{C} - \{0\}$. We estimate

$$\begin{aligned}
|\Theta(\theta,\epsilon)| &= |R^n e^{\sqrt{-1}n\theta} + (1-\epsilon)\{a_{n-1}R^{n-1}e^{i(n-1)\theta)} + \cdots + a_0\}|\\
&\ge R^n - \{|a_{n-1}|R^{n-1} + \cdots + |a_0|\} \ge R^n - R^{n-1}\{|a_{n-1}| + \cdots + |a_0|\}\\
&= R^{n-1}\{R - |a_{n-1}| + \cdots + |a_0|\} \ge 72R^{n-1} > 0\,.
\end{aligned}$$

Let $\tilde\gamma := \Theta(\theta, 1) = R^n e^{\sqrt{-1}n\theta}$. We then have $W(\tilde\gamma) = W(\gamma_R) = 0$. We have $\tilde\gamma(\theta) = R^n(\cos(n\theta) + \sqrt{-1}\sin(n\theta))$ so

$$\begin{aligned}
W(\tilde\gamma) &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{R^{2n}\cos(n\theta)n\sin(n\theta) - R^{2n}\sin(n\theta)(-n\sin(n\theta))}{R^{2n}\cos^2(n\theta) + R^{2n}\sin^2(n\theta)} d\theta\\
&= \frac{1}{2\pi}\int_{\theta=0}^{2\pi} n d\theta = n\,.
\end{aligned}$$

This is not zero; this contradiction establishes the desired result.