1. Differential Geometry quals Problems for Winter 2019

**Problem 1.1.** Let $\omega$ be a smooth $1$-form and let $X$ and $Y$ be smooth vector fields. Prove that $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$

**Solution.** Fix $\omega$. Let $T(X,Y) = \{d\omega\}(X,Y) - \left( X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \right)$.

We note that $\omega$ and $d\omega$ are tensors. 

\[
T(fX,Y) = \{d\omega\}(X,Y) - \left( X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \right)
\]

\[
= f\{d\omega\}(X,Y) - fX(\omega(Y)) + Y(f\omega(X)) + \omega(f[X,Y] - Y(f)X)
\]

\[
= f\{d\omega\}(X,Y) - fX(\omega(Y)) + Y(f\omega(X)) + f\omega([X,Y]) - fY(\omega(X))
\]

\[
= fT(X,Y).
\]

Since $T(X,Y) = -\mathcal{T}(Y,X)$, we also have $\mathcal{T}(X,fY) = fT(X,Y)$. Thus $\mathcal{T}$ is a tensor. Thus to show $\mathcal{T}$ vanishes, it suffices to check it on a coordinate frame $\partial_{x^i}$.

Let $\omega = a_i dx^i$. We compute 

\[
\omega \left( \partial_{x^j} \right) \left( \partial_{x^k} \right) = \partial_{x^j} a_k - \partial_{x^k} a_j = \partial_{x^j} a_k - \partial_{x^k} a_j,
\]

\[
X(\omega(Y)) = Y(\omega(X)) - \omega([X,Y]) = \partial_{x^k} a_j - \partial_{x^j} a_k = 0.
\]

**Problem 1.2.** Let $T$ be an isometry of a compact Riemannian manifold $(M,g)$. Let $S$ be a connected component of the fixed point set of $T$. Show $S$ is a smooth submanifold of $M$ which is totally geodesic.

**Solution.** Let $P \in S$. Let $T := \{ \xi \in TP : T_\xi \xi = \xi \}$. This is a linear subspace of $TP_M$. There exists $\epsilon > 0$ so that $\exp_P : \{ \xi \in TP_M : ||\xi|| < \epsilon \} \rightarrow M$ is a diffeomorphism onto its image $B_\epsilon(P)$. $\epsilon$ can also be chosen so that if $d(P,Q) < \epsilon$, then there is a unique geodesic from $P$ to $Q$ lying in $B_\epsilon(P)$. Thus $\exp_P(T \cap B_\epsilon(0))$ is a smooth diffeomorphism to its range $\tilde{S}$. If $\xi \in T$, then the curve $\exp_P(t\xi)$ is a geodesic in $M$. Since the initial point and initial direction of this geodesic is fixed by $T$, the whole geodesic is fixed by $T$ and hence lies in $S$. Thus $\tilde{S} \subseteq S$. On the other hand, if $Q \in B_\epsilon(P)$ and $TQ = Q$, then there is a unique geodesic from $P$ to $Q$ and hence this geodesic must be fixed by $T$. Thus the initial direction of this geodesic must be fixed and hence the initial direction lies in $T$. Thus $Q$ is in $\tilde{S}$. Thus $S = \tilde{S}$ near $P$ and $S$ is totally geodesic.

**Problem 1.3.** Let $O$ be a connected open subset of $\mathbb{R}^2$. Let $f$ be a smooth function on $O$ and let $ds^2 = e^{2f}(dx^2 + dy^2)$. Determine the scalar curvature $\tau$. Use the resulting formula to find the Gaussian curvature of the upper half plane with the metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$.

**Solution.** Suppose $ds^2 = e^{2f}(dx^2 + dy^2)$. We use the Koszul formula to express 

\[
g(\nabla_{\partial_x} \partial_x, \partial_y) = f_x e^{2f}, \quad g(\nabla_{\partial_x} \partial_y, \partial_x) = -f_y e^{2f}, \quad \nabla_{\partial_y} \partial_x = f_x \partial_x - f_y \partial_y,
\]

\[
g(\nabla_{\partial_x} \partial_y, \partial_y) = f_y e^{2f}, \quad \nabla_{\partial_y} \partial_y = \nabla_{\partial_x} \partial_x = f_y \partial_x + f_x \partial_y, \quad \nabla_{\partial_x} \partial_y = -f_x e^{2f}, \quad \nabla_{\partial_y} \partial_y = -f_x \partial_x + f_y \partial_y.
\]

We compute 

\[
(\nabla_{\partial_x} \nabla_{\partial_y} - \nabla_{\partial_y} \nabla_{\partial_x} - \nabla_{[\partial_x, \partial_y]} ) \partial_y
\]

\[
= \nabla_{\partial_x} \left( -f_x \partial_x + f_y \partial_y \right) - \nabla_{\partial_y} \left( f_x \partial_x + f_y \partial_y \right)
\]

\[
= -f_{xx} \partial_x - f_x \left( f_x \partial_x - f_y \partial_y \right) + f_{xy} \partial_y + f_y \left( f_y \partial_x + f_x \partial_y \right)
\]

\[
= -f_{yy} \partial_x - f_y \left( f_y \partial_x + f_x \partial_y \right) - f_x \left( -f_x \partial_x + f_y \partial_y \right)
\]

\[
= \left( -f_{xx} - f_{yy} - f_xx + f_yf_y - f_yf_y + f_xf_x \right) \partial_x
\]

\[
+ \left( f_xf_y + f_xy + f_yf_x - f_yf_y - f_xf_x \right) \partial_y
\]

\[
= \left( -f_{xx} + f_{yy} \right) \partial_x.
\]
We then have\[
\tau = 2 \frac{g(R(\partial_x, \partial_y)g(\partial_y, \partial_x))}{g(\partial_x, \partial_y)g(\partial_y, \partial_x) - g(\partial_x, \partial_y)^2} = -2e^{-2f}(f_{xx} + f_{yy}).\]
Suppose \(ds^2 = \frac{dx^2 + dy^2}{y^2}\). Then \(e^{2f} = y^2\) so \(2f = -2\log(y)\) and \(f = -\log(y)\). We have \(f_{xx} + f_{yy} = -2e^{-2f}(f_{xx} + f_{yy}) = -2y^2(y^{-2}) = -2\). That is correct.

**Problem 1.4.** Let \(M\) be a Riemannian manifold.

1. Let \(\{e_1, \ldots, e_m\}\) be a local orthonormal frame for the tangent bundle. Let \(\nabla\) be the Levi-Civita connection. Determine constants \(\{a, b, c\}\) so that\[
g(\nabla e_i, e_j, e_k) = a g([e_i, e_j], e_k) + b g([e_j, e_k], e_i) + c g([e_k, e_i], e_j).\]
2. Show that there exists a local orthonormal frame field with \([e_i, e_j] = 0\) for all \(i, j\) if and only if the curvature tensor vanishes identically.

**Solution.** We compute:\[
g(\nabla e_i, e_j, e_k) = g([e_i, e_j], e_k) + g(\nabla e_i e_j, e_k) \text{ (torsion free)}
\]
\[
= g(e_i e_j, e_k) + c g(e_i, e_k) - g(e_i, \nabla e_j e_k) \text{ (Riemannian)}
\]
\[
= g(e_i e_j, e_k) - g(e_i, e_j e_k) - g(e_i, \nabla e_k e_j) \text{ (torsion free)}
\]
\[
= g([e_i, e_j], e_k) + g([e_k, e_i], e_j) - g(\nabla e_i e_j, e_k) \text{ (Riemannian)}
\]
\[
= g([e_i, e_j], e_k) - g([e_k, e_i], e_j) + g(\nabla e_i e_j, e_k) \text{ (torsion free)}
\]
\[
= g([e_i, e_j], e_k) - g([e_i, e_k], e_j) - g(e_i, e_k, e_j) - g(e_k, \nabla e_i, e_j) + 2g(\nabla e_i, e_k) = g(e_i, [e_i, e_j]) + g([e_k, e_i], e_i) + g([e_i, e_j], e_k).
\]

So \(a = \frac{1}{2}, b = -\frac{1}{2}\), and \(c = \frac{1}{2}\).

If the curvature tensor vanishes identically, then \(M\) has constant sectional curvature \(0\). So \(M\) is locally isometric to flat space. The coordinates frame satisfies \(g(\partial_{x_i}, \partial_{x_j}) = \delta_{ij}\) and \([\partial_{x_i}, \partial_{x_j}] = 0\). This provides the requisite torsion free orthonormal basis. Conversely, if such an orthonormal basis exists, then we have that \(g(\nabla e_i, e_j, e_k) = 0\) for all \(i, j, k\). This implies \(\nabla e_i, e_j = 0\) for all \(i, j\). Expand \([e_i, e_j] = C_{ij}^k e_k\). Then
\[
R(e_i, e_j) e_k = \nabla e_i \nabla e_j e_k - \nabla e_j \nabla e_i e_k - \nabla [e_i, e_j] e_k = \nabla e_i 0 - \nabla e_j 0 - C_{ij}^k \nabla e_i e_k = 0.
\]

**Problem 1.5.** Let \(\mathbb{Z}_f = \{\lambda \in \mathbb{C} : \lambda^f = 1\}\) be the cyclic group of \(f\)th roots of unity. Let \(S^{2n-1} := \{z = (z_1, \ldots, z^n) \in \mathbb{C}^n : |z|^2 = 1\}\) be the unit sphere. Let \(\mathbb{Z}_f\) act on \(S^{2n-1}\) by complex multiplication. The quotient \(M_{n,f} := S^{2n-1}/\mathbb{Z}_f\) is called a classical or diagonal lens space. It is a smooth manifold - you need not prove this. Determine the de Rham cohomology of \(M_{n,f}\).

**Solution.** Let \(\Gamma\) be a finite group acting by smooth isometries and without fixed points on a compact manifold \(M\). We showed that \(H^p_{\text{DeR}}(M/\Gamma) = H^p_{\text{DeR}}(M)\) is the space of invariant cohomology classes. Let \(\lambda(t) = e^{2\pi \sqrt{-1} t}\) act on \(S^{2n-1}\) by complex multiplication to define a smooth 1-parameter family of diffeomorphisms of \(S^{2n-1}\). Since \(\lambda(0) = 1\) is the the identity, \(\lambda(t)\) is the identity acting on \(H^p_{\text{DeR}}(S^{2n-1})\) for any \(p\).

Thus \(H^p_{\text{DeR}}(S^{2n-1}/\mathbb{Z}_f)\) is isomorphic to \(H^p_{\text{DeR}}(S^{2n-1})\) for any \(p\). This is \(\mathbb{R}\) if \(p = 0\) or if \(p = 2n - 1\); it is zero otherwise.

**Problem 1.6.** A 3-dimensional connected Lie group \(G\) has a Lie algebra \(\mathfrak{g}\) where relative to a suitable basis we have \([e_1, e_3] = e_1\) and \([e_2, e_3] = -e_2\). Determine the associated Lie algebra cohomology. Can \(G\) be compact? Justify your answer.

**Solution:** We use the formula \(d\omega(e_a \wedge e_b) = -\omega([e_a, e_b])\) to compute:
\[
d^1 = 0, \quad d e^1 = -e^1 \wedge e^3, \quad d e^2 = e^2 \wedge e^3, \quad d e^3 = 0,
\]
\[
d(e^1 \wedge e^2) = -e^1 \wedge e^3 \wedge e^2 - e^1 \wedge e^2 \wedge e^3 = 0, \quad d(e^2 \wedge e^3) = 0, \quad d(e^1 \wedge e^3) = 0,
\]
\[
d(e^1 \wedge e^2 \wedge e^3) = 0.
\]
Consequently $H^*(g) = \mathbb{R} \cdot \mathbb{I} \oplus \mathbb{R} \cdot e^3 \oplus \mathbb{R} \cdot e^1 \wedge e^2 \oplus \mathbb{R} \cdot e^1 \wedge e^2 \wedge e^3$. If there exists a compact group $G$ whose Lie algebra is $g$, then $H^*(G) = H^*(g)$. Furthermore, $H^*(G)$ is an exterior algebra on odd dimensional generators; in particular $H^2(G)$ is generated by $H^1(G)$. This is not the case for $H^*(g)$ and hence no such compact group exists.

**Problem 1.7.** Show that any complex manifold is orientable.

**Solution.** The following probably is overkill and I would accept a solution which simply assumed the result from linear algebra. We begin with a result from linear algebra. We may identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. This identification defines an embedding $\iota : \text{GL}(n, \mathbb{C}) \to \text{GL}(2n, \mathbb{R})$. If $T \in \text{GL}(n, \mathbb{C})$, show that $|\det(T)|^2 = |\det(\iota(T))|^2$. Suppose first $n = 1$. Let $e_1$ be the standard basis for $\mathbb{C}$ so $\{e_1, \sqrt{-1}e_1\}$ is the corresponding basis for $\mathbb{R}^2$. Then

$$T(e_1) = a_1 e_1 + \sqrt{-1}a_2 e_1$$

and $|\det(T)|^2 = a_1^2 + a_2^2$.

Suppose next $n$ is arbitrary and $T = \text{diag}(\alpha_1, \ldots, \alpha_n)$ is diagonal. Then we establish the desired identity by computing

$$|\det(T)|^2 = |\det(\text{diag}(\alpha_1 \ldots \alpha_n))|^2 = |\alpha_1 \ldots \alpha_n|^2 = |\alpha_1|^2 \ldots |\alpha_n|^2$$

$$\text{det}(\iota(T)) = \text{det}(\iota(\alpha_1 \text{id}) \oplus \cdots \oplus \iota(\alpha_n \text{id})) = \text{det}(\iota(\alpha_1 \text{id})) \cdots \text{det}(\iota(\alpha_n \text{id})) = |\alpha_1|^2 \ldots |\alpha_n|^2.$$

Suppose next $n$ is arbitrary and $T$ is diagonalizable so there exists $g \in \text{GL}(n, \mathbb{C})$ so that $T = g \text{diag}(\alpha_1, \ldots, \alpha_n)g^{-1}$. We compute

$$|\det(T)|^2 = |\det(gTg^{-1})|^2 = |\alpha_1 \ldots \alpha_n|^2 = |\alpha_1|^2 \ldots |\alpha_n|^2$$

$$= \text{det}(gTg^{-1}) = \text{det}(\iota(T)g(t)g^{-1}) = |\det(\iota(T))|^2.$$

This is where I would expect the average student to start If $M$ is holomorphic, the transition functions $\Phi_{\alpha,\beta}$ are holomorphic and thus $d\Phi_{\alpha,\beta} \in \text{GL}(n, \mathbb{C})$. The corresponding real transition functions are then given by $\iota \{d\Phi_{\alpha,\beta}\}$. Thus we have $\text{det}(\iota(\Phi_{\alpha,\beta})) = |\text{det}(d\Phi_{\alpha,\beta})| > 0$ and $M$ is orientable.

**Problem 1.8.** Let $\mathbb{RP}^m = S^m/\mathbb{Z}_2$ be the real projective space. Let $L$ be the real line bundle over $\mathbb{RP}^m$ defined by taking $(S^m \times \mathbb{R})/\mathbb{Z}_2$ where we identify $(\bar{x}, \lambda)$ with $(-\bar{x}, -\lambda)$ for $\bar{x} \in S^m$ and $\lambda \in \mathbb{R}$. Let $\mathbb{I} = \mathbb{RP}^m \times \mathbb{R}$ be the trivial real line bundle over $\mathbb{RP}^m$ and let $T(\mathbb{RP}^m)$ be the tangent bundle of $\mathbb{RP}^m$. Construct an isomorphism between the stable tangent bundle $T(\mathbb{RP}^m) \oplus \mathbb{I}$ and the direct sum of $(m+1)$ copies of $L$.

**Solution.** Let $\nu$ be the unit normal to the sphere; $\nu(\bar{x}) = \bar{x}$ for $\bar{x} \in S^m \subset \mathbb{R}^{m+1}$. Let $\xi \in T(S^m)$. Then $T(S^m) = \{(\bar{x}, \ell) \in S^m \times \mathbb{R}^{m+1} : \bar{x} \perp \ell\}$ and

$$T(S^m) \oplus 1_S^m = \{(\bar{x}, \ell, \lambda) : \bar{x} \perp \lambda, \lambda \in \mathbb{R}\}$$

$$\mathbb{I}^{m+1}_S = \{(\bar{x}, \bar{s}) : \bar{x} \in S^m \text{ and } \bar{s} \in \mathbb{R}^{m+1}\}.$$

Define an isomorphism $\Phi$ from $T(S^m) \oplus 1_S^m$ to $\mathbb{I}^{m+1}_S$ by setting

$$\Phi(\bar{x}, \ell, \lambda) = (\bar{x}, \ell + \lambda \bar{x}).$$

Note that $\Phi(-\bar{x}, -\ell, \lambda) = -\Phi(\bar{x}, \ell, \lambda)$. Thus $\Phi$ descends to an isomorphism between $T(\mathbb{RP}^m) \oplus 1_{\mathbb{RP}^m}$ and $\mathbb{I}^{m+1}$. 
Problem 1.9. Let \( T(\theta_1, \theta_2) = (\cos(\theta_1)(\cos(\theta_2) + 2), \sin(\theta_1)(\cos(\theta_2) + 2), \sin(\theta_2)) \) for \( 0 \leq \theta_1 \leq 2\pi \) and \( 0 \leq \theta_2 \leq 2\pi \) parametrize a toroidal surface of revolution. Determine the first fundamental form I, the second fundamental form II, the principal curvatures, and the Gauss curvature.

Solution.

\[
\begin{align*}
\partial_{\theta_1} T &= (\cos(\theta_2) + 2)(-\sin(\theta_1), \cos(\theta_1), 0), \\
\partial_{\theta_2} T &= (-\sin(\theta_2) \cos(\theta_1), -\sin(\theta_2) \sin(\theta_1), \cos(\theta_2)), \\
g(\partial_{\theta_1}, \partial_{\theta_2}) &= (\cos(\theta_2) + 2)^2, \\
\partial_{\theta_1} \times \partial_{\theta_2} &= (\cos(\theta_2) + 2)(\cos(\theta_1) \cos(\theta_2), \sin(\theta_1) \cos(\theta_2), \sin(\theta_2)), \\
N &= (\cos(\theta_1) \cos(\theta_2), \sin(\theta_1) \cos(\theta_2), \sin(\theta_2)), \\
\partial_{\theta_1} \partial_{\theta_1} T &= (\cos(\theta_2) + 2)(-\cos(\theta_1), -\sin(\theta_1), 0), \\
\partial_{\theta_1} \partial_{\theta_2} T &= -\sin(\theta_2)(-\sin(\theta_1), \cos(\theta_1), 0), \\
\partial_{\theta_2} \partial_{\theta_2} T &= (-\cos(\theta_2) \cos(\theta_1), -\cos(\theta_2) \sin(\theta_1), -\sin(\theta_2)), \\
L_{11} &= \partial_{\theta_1} \partial_{\theta_1} T \cdot N = -(\cos(\theta_2) + 2) \cos(\theta_2), \\
L_{12} &= \partial_{\theta_1} \partial_{\theta_2} T \cdot N = 0, \\
L_{22} &= \partial_{\theta_2} \partial_{\theta_2} T \cdot N = -1,
\end{align*}
\]

\[
\kappa_1 = L_{11}/g_{11} = -\frac{\cos(\theta_2)}{\cos(\theta_2) + 2}, \quad \kappa_2 = -1, \quad \kappa = \frac{\cos(\theta_2)}{\cos(\theta_2) + 2}.
\]

Problem 1.10. Use Stokes Theorem to prove the fundamental theorem of algebra.

Solution Let \( p(z) = a_n z^n + \cdots + a_0 \) be a complex polynomial where \( n \geq 1 \) and \( a_n \neq 0 \). Assume \( p(z) \neq 0 \) for all \( z \in \mathbb{C} \) and argue at length for a contradiction. Without loss of generality, we may assume \( a_n = 1 \). If \( \gamma : S^1 \to \mathbb{C} - \{0\} \), define

\[
W(\gamma) = \frac{1}{2\pi} \int_{\gamma} \omega \quad \text{for} \quad \omega := \frac{x \, dy - y \, dx}{x^2 + y^2}.
\]

Verify \( d\omega = 0 \). Let \( \Gamma : S^1 \times [0,1] \to \mathbb{C} - \{0\} \). Let \( \gamma_i(\theta) = \Gamma(\theta, i) \). Since \( d\omega = 0 \), we can use Stoke's theorem to see

\[
0 = \int_{S^1 \times [0,1]} \Gamma^* d\omega = \int_{S^1} d\Gamma^* \omega
= \int_{S^1 \times \{1\}} \gamma_1^* \omega - \int_{S^1 \times \{0\}} \gamma_0^* \omega = 2\pi \{ W(\gamma_1) - W(\gamma_0) \}.
\]

Let \( \Gamma(\theta, r) = p(re^{i\theta}) \). Then \( \Gamma_0 \) is the trivial path. Thus \( W(\gamma_0) = 0 \). Let \( R = |a_{n-1}| + \cdots + |a_0| + 72; W(\Gamma_R) = W(\gamma_0) = 0 \). For \( \epsilon \in [0,1] \), define

\[
\Theta(\theta, \epsilon) = \epsilon R^n e^{\epsilon n \theta} + (1 - \epsilon)p(R e^{\epsilon n \theta}).
\]

We must show \( \Theta \) takes values in \( \mathbb{C} - \{0\} \). We estimate

\[
|\Theta(\theta, \epsilon)| &= |R^n e^{\epsilon n \theta} + (1 - \epsilon)|\{a_{n-1}R^{n-1} e^{(n-1)\theta} + \cdots + a_0| \\
&\geq R^n - \{ |a_{n-1}|R^{n-1} + \cdots + |a_0| \} \geq R^n - R^{n-1}\{ |a_{n-1}| + \cdots + |a_0| \} \\
&= R^{n-1}\{ R - |a_{n-1}| + \cdots + |a_0| \} \geq 72 R^{n-1} > 0.
\]

Let \( \tilde{\gamma} := \Theta(\theta, 1) = R^n e^{n \theta} \). We then have \( W(\tilde{\gamma}) = W(\gamma_R) = 0 \). We have \( \tilde{\gamma}(\theta) = R^n (\cos(n \theta) + \sqrt{-1} \sin(n \theta)) \) so

\[
W(\tilde{\gamma}) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} R^{2n} \cos(n \theta) n \sin(n \theta) - R^{2n} \sin(n \theta) (-n \sin(n \theta)) d\theta = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} n d\theta = n.
\]

This is not zero; this contradiction establishes the desired result.