1. Differential Geometry quals Problems for Winter 2019

Problem 1.1. Let ω be a smooth 1-form and let X and Y be smooth vector fields. Prove that $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$.

Problem 1.2. Let T be an isometry of a compact Riemannian manifold (M,g). Let S be a connected component of the fixed point set of T. Show S is a smooth submanifold of M which is totally geodesic.

Problem 1.3. Let $\mathcal{O}$ be a connected open subset of $\mathbb{R}^2$. Let f be a smooth function on $\mathcal{O}$ and let $ds^2 = e^{2f}(dx^2 + dy^2)$. Determine the scalar curvature τ. Use the resulting formula to find the Gaussian curvature of the upper half plane with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$.

Problem 1.4. Let $\mathcal{M}$ be a Riemannian manifold.

(1) Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame for the tangent bundle. Let ∇ be the Levi-Civita connection. Determine constants $\{a,b,c\}$ so that $g(\nabla_{e_i} e_j, e_k) = a * g([e_i, e_j], e_k) + b * g([e_j, e_k], e_i) + c * g([e_k, e_i], e_j)$.

(2) Show that there exists a local orthonormal frame field with $[e_i, e_j] = 0$ for all i, j if and only if the curvature tensor vanishes identically.

Problem 1.5. Let $\mathbb{Z}_\ell = \{\lambda \in \mathbb{C} : \lambda^\ell = 1\}$ be the cyclic group of ℓ^{th} roots of unity. Let $S^{2n-1} := \{\vec{z} = (z^1, \ldots, z^n) \in \mathbb{C}^n : |\vec{z}|^2 = 1\}$ be the unit sphere. Let $\mathbb{Z}_\ell$ act on S^{2n-1} by complex multiplication. The quotient $M_{n,\ell} := S^{2n-1}/\mathbb{Z}_\ell$ is called a classical or diagonal lens space. It is a smooth manifold - you need not prove this. Determine the de Rham cohomology of $M_{n,\ell}$.

Problem 1.6. A 3-dimensional connected Lie group G has a Lie algebra $\mathfrak{g}$ where relative to a suitable basis we have $[e_1, e_3] = e_1$ and $[e_2, e_3] = -e_2$. Determine the associated Lie algebra cohomology. Can G be compact? Justify your answer.

Problem 1.7. Show that any complex manifold is orientable.

Problem 1.8. Let $\mathbb{RP}^m = S^m/\mathbb{Z}_2$ be real projective space. Let $\mathbb{L}$ be the real line bundle over $\mathbb{RP}^m$ defined by taking $(S^m \times \mathbb{R})/\mathbb{Z}_2$ where we identify $(\vec{x}, \lambda)$ with $(-\vec{x}, -\lambda)$ for $\vec{x} \in S^m$ and $\lambda \in \mathbb{R}$. Let $\mathbb{1} = \mathbb{RP}^m \times \mathbb{R}$ be the trivial real line bundle over $\mathbb{RP}^m$ and let $T(\mathbb{RP}^m)$ be the tangent bundle of $\mathbb{RP}^m$. Construct an isomorphism between the stable tangent bundle $T(\mathbb{RP}^m) \oplus \mathbb{1}$ and the direct sum of $(m+1)$ copies of $\mathbb{L}$.

Problem 1.9. Let $T(\theta_1, \theta_2) = (\cos(\theta_1)(\cos(\theta_2) + 2), \sin(\theta_1)(\cos(\theta_2) + 2), \sin(\theta_2))$ for $0 \le \theta_1 \le 2\pi$ and $0 \le \theta_2 \le 2\pi$ parametrize a toroidal surface of revolution. Determine the first fundamental form I, the second fundamental form II, the principal curvatures, and the Gauss curvature.

Problem 1.10. Use Stokes Theorem to prove the fundamental theorem of algebra.