Problem 1.1. Let $\omega$ be a smooth 1-form and let $X$ and $Y$ be smooth vector fields. Prove that $d\omega(X,Y) = \omega(Y) - \omega(X) - \omega([X,Y])$.

Problem 1.2. Let $T$ be an isometry of a compact Riemannian manifold $(M,g)$. Let $S$ be a connected component of the fixed point set of $T$. Show $S$ is a smooth submanifold of $M$ which is totally geodesic.

Problem 1.3. Let $O$ be a connected open subset of $\mathbb{R}^2$. Let $f$ be a smooth function on $O$ and let $ds^2 = e^{2f}(dx^2 + dy^2)$. Determine the scalar curvature $\tau$. Use the resulting formula to find the Gaussian curvature of the upper half plane with the metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$.

Problem 1.4. Let $\mathcal{M}$ be a Riemannian manifold.

1. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame for the tangent bundle. Let $\nabla$ be the Levi-Civita connection. Determine constants $\{a,b,c\}$ so that $g(\nabla e_i, e_j, e_k) = a * g([e_i, e_j], e_k) + b * g([e_j, e_k], e_i) + c * g([e_k, e_i], e_j)$.

2. Show that there exists a local orthonormal frame field with $[e_i, e_j] = 0$ for all $i, j$ if and only if the curvature tensor vanishes identically.

Problem 1.5. Let $\mathbb{Z}_\ell = \{\lambda \in \mathbb{C} : \lambda^\ell = 1\}$ be the cyclic group of $\ell^{th}$ roots of unity. Let $S^{2n-1} := \{\bar{z} = (z^1, \ldots, z^n) \in \mathbb{C}^n : |\bar{z}|^2 = 1\}$ be the unit sphere. Let $\mathbb{Z}_\ell$ act on $S^{2n-1}$ by complex multiplication. The quotient $M_{n,\ell} := S^{2n-1}/\mathbb{Z}_\ell$ is called a classical or diagonal lens space. It is a smooth manifold - you need not prove this. Determine the de Rham cohomology of $M_{n,\ell}$.

Problem 1.6. A 3-dimensional connected Lie group $G$ has a Lie algebra $\mathfrak{g}$ where relative to a suitable basis we have $[e_1, e_3] = e_1$ and $[e_2, e_3] = -e_2$. Determine the associated Lie algebra cohomology. Can $G$ be compact? Justify your answer.

Problem 1.7. Show that any complex manifold is orientable.

Problem 1.8. Let $\mathbb{RP}^m = S^m/\mathbb{Z}_2$ be real projective space. Let $L$ be the real line bundle over $\mathbb{RP}^m$ defined by taking $(S^m \times \mathbb{R})/\mathbb{Z}_2$ where we identify $(\bar{x}, \lambda)$ with $(-\bar{x}, -\lambda)$ for $\bar{x} \in S^m$ and $\lambda \in \mathbb{R}$. Let $\mathbb{I} = \mathbb{RP}^m \times \mathbb{R}$ be the trivial real line bundle over $\mathbb{RP}^m$ and let $T(\mathbb{RP}^m)$ be the tangent bundle of $\mathbb{RP}^m$. Construct an isomorphism between the stable tangent bundle $T(\mathbb{RP}^m) \oplus \mathbb{I}$ and the direct sum of $(m+1)$ copies of $L$.

Problem 1.9. Let $T(\theta_1, \theta_2) = (\cos(\theta_1)(\cos(\theta_2) + 2), \sin(\theta_1)(\cos(\theta_2) + 2), \sin(\theta_2))$ for $0 \leq \theta_1 \leq 2\pi$ and $0 \leq \theta_2 \leq 2\pi$ parametrize a toroidal surface of revolution. Determine the first fundamental form $I$, the second fundamental form $II$, the principal curvatures, and the Gauss curvature.

Problem 1.10. Use Stokes Theorem to prove the fundamental theorem of algebra.