SOLUTIONS FOR ANALYSIS QUALIFYING EXAM, WINTER 2018

Notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$. (That is, $0 \notin \mathbb{N}$.)

1. Suppose f is a Lebesgue measurable function on $\mathbb{R}$ such that f and $x \mapsto xf(x)$ are both in $L^2(\mathbb{R})$. Prove that $f \in L^1(\mathbb{R})$.

Solution. We divide $\mathbb{R}$ into two parts, $A = [-1, 1]$ and its complement $\mathbb{R} \setminus A$. On A, by Hölder's inequality we have

$$\left(\int_A |f| \, dm \right)^2 \leq 2 \int_A |f|^2 \, dm < \infty.$$

On $\mathbb{R} \setminus A$, by Hölder's inequality we have

$$\left(\int_{\mathbb{R} \setminus A} |f| \, dm \right)^2 \leq \left(\int_{\mathbb{R} \setminus A} x^{-2} \, dm(x) \right) \left(\int_{\mathbb{R} \setminus A} x^2 |f(x)|^2 \, dm(x) \right) < \infty.$$

Therefore

$$\int_{\mathbb{R}} |f| \, dm = \int_A |f| \, dm + \int_{\mathbb{R} \setminus A} |f| \, dm < \infty,$$

as desired. □

2. Let f be a nonnegative Lebesgue measurable function on $[0, 1]$ such that $f > 0$ almost everywhere. Prove that for any $\epsilon > 0$ there is $\delta > 0$ such that for any Lebesgue measurable subset E of $[0, 1]$ with $m(E) > \epsilon$, we have $\int_E f \, dm > \delta$.

Solution. Let $\epsilon > 0$. For $n \in \mathbb{N}$, define $F_n = \{x \in [0, 1] \colon f(x) > 1/n\}$. Then $(F_n)_{n \in \mathbb{N}}$ is an increasing sequence of measurable sets. Since $f > 0$ holds almost everywhere, we have $m\left(\bigcup_{n=1}^{\infty} F_n\right) = 1$. This implies that $\lim_{n \to \infty} m(F_n) = 1$. Therefore there is $n_0 \in \mathbb{N}$ such that $m(F_{n_0}) > 1 - \epsilon/2$. Define $\delta = \epsilon/(2n_0)$.

Now let $E \subset [0, 1]$ be a measurable set such that $m(E) > \epsilon$. Then

$$m(F_{n_0} \cap E) = m(F_{n_0}) + m(E) - m(F_{n_0} \cup E) \geq m(F_{n_0}) + \epsilon - 1.$$

Hence

$$m(F_{n_0} \cap E) > \frac{\epsilon}{2}.$$

Therefore

$$\int_E f \, dm \geq \int_{F_{n_0} \cap E} f \, dm \geq \frac{1}{n_0} m(F_{n_0} \cap E) > \frac{\epsilon}{2n_0} = \delta.$$

This completes the solution. □

3. Let m be Lebesgue measure on $[0, 1]$. Suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence of strictly positive measurable functions on $[0, 1]$ such that

$$\lim_{n \to \infty} \int_{[0,1]} f_n^2 \, dm = 0.$$

Prove that

$$\lim_{n \to \infty} \int_{[0,1]} |f_n \log(f_n)| \, dm = 0.$$

Solution. Let $\epsilon > 0$. Since $\lim_{y \to 0^+} y \log(y) = 0$, there is $\delta > 0$ such that $|y \log(y)| < \epsilon/2$ for $y \in (0, \delta)$.

Define $C = \max\left(1, \, -\delta^{-1} \log(\delta)\right)$. We claim that $|\log(y)| \leq Cy$ for all $y \in [\delta, \infty)$. Certainly if $y \geq 1$, then $0 \leq \log(y) < y \leq Cy$. If $\delta \geq 1$, we are done. If $0 < \delta < 1$, then log is negative and increasing on $[\delta, 1]$, so for $y \in [\delta, 1]$ we have

$$|\log(y)| \leq -\log(\delta) \leq \left(-\frac{\log(\delta)}{\delta} \right) y \leq Cy.$$

The claim is proved.

By hypothesis, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\int_{[0,1]} f_n^2 \, dm < \frac{\epsilon}{2C}.$$

Let $n \geq N$. Set $E = \{x \in [0,1] \colon f(x) < \delta\}$. Then

$$\begin{aligned}
\int_{[0,1]} |f_n \log(f_n)| \, dm &= \int_E |f_n \log(f_n)| \, dm + \int_{[0,1]\setminus E} |f_n \log(f_n)| \, dm \\
&\leq \left(\frac{\epsilon}{2}\right) m(E) + \int_{[0,1]\setminus E} C f_n^2 \, dm \\
&\leq \frac{\epsilon}{2} + C \int_{[0,1]} f_n^2 \, dm < \frac{\epsilon}{2} + C\left(\frac{\epsilon}{2C}\right) = \epsilon.
\end{aligned}$$

This completes the solution. $\square$

The alternate solution below is different in two ways. It doesn't use sequences, and it uses a different arrangement of estimates. These two changes are independent of each other.

Alternate solution. It is enough to show that for all $\epsilon > 0$ there is $\delta > 0$ such that whenever $f \colon [0,1] \to (0,\infty)$ is measurable and $\int_{[0,1]} f^2 \, dm < \delta$, then $\int_{[0,1]} |f \log(f)| \, dm < \epsilon$.

Begin by observing that $\lim_{y \to 0^+} y \log(y) = 0$. Since $y \mapsto y \log(y)$ is continuous on $(0,1]$, it follows that there is $M \in (0,\infty)$ such that $|y \log(y)| \leq M$ for all $y \in (0,1]$. (In fact, methods of elementary calculus show that for all $y \in (0,1]$, we have $0 \geq y \log(y) \geq -\frac{1}{e}$. But we don't need the exact bound.)

Using $\lim_{y \to 0^+} y \log(y) = 0$, choose $\rho > 0$ such that $|y \log(y)| < \frac{\epsilon}{3}$ for all $y \in (0,\rho)$. Then set

$$\delta = \min\left(\frac{\epsilon}{3}, \frac{\epsilon\rho^2}{3M}\right).$$

Now suppose that $f \colon [0,1] \to (0,\infty)$ is measurable and $\int_{[0,1]} f^2 \, dm < \delta$. Define

$$E_1 = \{x \in [0,1] \colon f(x) < \rho\}, \qquad E_2 = \{x \in [0,1] \colon \rho \leq f(x) \leq 1\}, \qquad \text{and} \qquad E_3 = \{x \in [0,1] \colon 1 < f(x)\}.$$

Then

$$\int_{E_1} |f \log(f)| \, dm \leq \left(\frac{\epsilon}{3}\right) m(E_1) \leq \frac{\epsilon}{3} \tag{1}$$

and, since $0 < \log(f(x)) < f(x)$ for $x \in E_3$,

$$\int_{E_3} |f \log(f)| \, dm \leq \int_{E_3} f^2 \, dm \leq \int_{[0,1]} f^2 \, dm < \delta \leq \frac{\epsilon}{3}. \tag{2}$$

We claim that

$$m(E_2) \leq \frac{\epsilon}{3M}.$$

We have, using $f(x) \geq \rho$ for $x \in E_2$ at the fourth step,

$$\frac{\epsilon\rho^2}{3M} \geq \delta > \int_{[0,1]} f^2 \, dm \geq \int_{E_2} f^2 \, dm \geq \rho^2 m(E_2),$$

from which the claim follows. Now, using the claim and $|f(x) \log(f(x))| \leq M$ for $x \in E_2$,

$$\int_{E_2} |f \log(f)| \, dm \leq M m(E_2) < \frac{\epsilon}{3}.$$

Combining this estimate with (1) and (2), we get $\int_{[0,1]} |f \log(f)| \, dm < \epsilon$, as desired. $\square$

4. For which $p \in [1,\infty]$ is there a nonzero bounded linear functional T on $L^p([0,1])$ such that T vanishes on the subspace $C([0,1]) \subset L^p([0,1])$? Justify your answer.

Solution. If $p \in [1,\infty)$ then $C([0,1])$ is dense in $L^p([0,1])$. Hence any bounded linear functional on $L^p([0,1])$ which vanishes on $C([0,1])$ must be zero.

For $p = \infty$, we construct such a nonzero linear functional T. Fix any $a \in (0,1)$, for example, $a = \frac{1}{2}$. Let $h = \chi_{[0,a]} \in L^\infty([0,1])$. Let $M = \operatorname{span}(h, C([0,1]))$. Define $T_0 \colon M \to \mathbb{C}$ by

$$T_0(f) = \lim_{x \to a^+} f(x) - \lim_{x \to a^-} f(x)$$

for $h \in M$. Then T_0 is well defined because the left and right limits at a exist for all $f \in C([0,1])$ and also for $f = h$. Obviously T is linear. Also, for $f \in M$ we have

$$|T_0(f)| \leq \left|\lim_{x\to a^+} f(x)\right| + \left|\lim_{x\to a^-} f(x)\right| \leq \sup_{x\in[0,a]} |f(x)| + \sup_{x\in[a,1]} |f(x)| \leq 2\|f\|_{L^\infty([0,1])}.$$

So T_0 is bounded.

Clearly $T_0(f) = 0$ for all $f \in C([0,1])$ and $T_0(h) = -1$. By Hahn-Banach Extension Theorem, there is a bounded linear functional T on $L^\infty([0,1])$ which extends T_0. Then T is a nonzero bounded linear functional on $L^\infty([0,1])$ which vanishes on $C([0,1])$. □

5. Let l^1 be the Banach space of complex sequences defined as follows:

$$a = (a_1, a_2, a_3, \ldots) \in l^1 \text{ if and only if } \|a\|_{l^1} = \sum_{k=1}^{\infty} |a_k| \text{ is finite.}$$

Let X be a separable Banach space. Suppose $\{x_k : k \in \mathbb{N}\}$ is a countable dense subset of the closed unit ball in X. Define a linear operator $S\colon l^1 \to X$ by

$$Sa = \sum_{k=1}^{\infty} a_k x_k$$

for $a \in l^1$. Prove that S is bounded and surjective.

Solution. For $a \in l^1$, the series $\sum_{k=1}^{\infty} a_k x_k$ converges absolutely, because

$$\sum_{k=1}^{\infty} \|a_k x_k\|_X = \sum_{k=1}^{\infty} |a_k| \|x_k\|_X \leq \sum_{k=1}^{\infty} |a_k| = \|a\|_{l^1}.$$

Therefore $\sum_{k=1}^{\infty} a_k x_k$ converges. It follows that Sa is well defined, and that $\|Sa\|_X \leq \|a\|_{l^1}$. It is now immediate to check that S is linear.

The main point is to prove that S is surjective. We only need to show that any x in the closed unit ball of X is in the range of S. We construct by induction a sequence $l_1, l_2, \ldots \in \mathbb{N}$ such that $l_1 < l_2 < \cdots$ and such that

$$\left\| x - \sum_{j=1}^{n} 2^{1-j} x_{l_j} \right\| \leq 2^{-n} \tag{3}$$

for all $n \in \mathbb{N}$. Since $\{x_k : k \in \mathbb{N}\}$ is dense in the closed unit ball, there is $l_1 \in \mathbb{N}$ such that $\|x - x_{l_1}\| \leq 1/2$. Given l_n, we have

$$\left\| 2^n x - \sum_{j=1}^{n} 2^{n+1-j} x_{l_j} \right\| \leq 1.$$

Since $\{x_k : k > l_n\}$ is also dense in the closed unit ball, there is $l_{n+1} > l_n$ such that

$$\left\| 2^n x - \sum_{j=1}^{n} 2^{n+1-j} x_{l_j} - x_{l_{n+1}} \right\| \leq \frac{1}{2}.$$

Then

$$\left\| x - \sum_{j=1}^{n+1} 2^{1-j} x_{l_j} \right\| \leq 2^{-n-1}.$$

This completes the induction.

Now define

$$a_k = \begin{cases} 2^{-n} & n \in \mathbb{N} \text{ and } k = l_n \\ 0 & k \notin \{l_1, l_2, \ldots\}. \end{cases}$$

Then $a \in l^1$; indeed, $\|a\|_{l^1} = 1$. Since (3) holds for all $n \in \mathbb{N}$, we have $Sa = x$. □

Most of the credit is for the proof of surjectivity.

6. For $t \in \mathbb{R}$, let $T_t \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator $(T_t f)(x) = f(x-t)$ for $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. Prove that $\|T_t - T_s\| \geq 2$ for $t \neq s$, where the norm is the operator norm.

First we sketch the ideas behind the arguments below. Since Lebesgue measure m is translation invariant, for all for $t \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$ we have

$$\|T_t f\|^2 = \int_{\mathbb{R}} |T_t f|^2 \, dm = \int_{\mathbb{R}} |f(x-t)|^2 \, dm(x) = \int_{\mathbb{R}} |f(y)|^2 \, dm(y) = \|f\|^2.$$

Hence $\|T_t\| = 1$ for each t. The strategy is to construct functions $f_n \in L^2(\mathbb{R})$ such that $\|(T_t - T_s) f_n\|$ is close to $2\|f_n\|$. We compute, for each $f \in L^2(\mathbb{R})$,

$$\|(T_t - T_s) f\|^2 = \|T_t f\|^2 + \|T_s f\|^2 - 2\langle T_t f, \, T_s f \rangle.$$

Hence to achieve the above approximation for fixed distinct t and s, we need

$$\langle T_t f, \, T_s f \rangle = \int_{\mathbb{R}} f(x-t) \overline{f(x-s)} \, dm = \int_{\mathbb{R}} f(y) \overline{f(y-s+t)} \, dm$$

to be close to $-\|f\|^2$. With $a = t - s$, this means we want $f(y)$ to be close to $-f(y-a)$. With this in mind we are ready to construct a sequence which gives the actual proof.

Solution. By exchanging s and t, we can assume $s < t$. Set $a = t - s$. Define

$$f_n = \sum_{k=1}^{n} (-1)^k \chi_{[(k-1)a, \, ka)}.$$

(Thus f is supported on $[0, na)$, and alternates in sign on subintervals of length a. The function $x \mapsto f_n(x) + f_n(x-a)$ is supported on $[-a, 0) \cup [(n-1)a, \, na)$.)

Clearly $|f_n|^2 = \chi_{[0,na)}$ and hence $\|f_n\|^2 = na$. We compute directly

$$\|(T_t - T_s) f_n\|^2 = \|T_t(f_n)\|^2 + \|T_s(f_n)\|^2 - 2\langle T_t(f_n), \, T_s f_n \rangle = 2na - 2\langle T_t(f_n), \, T_s(f_n) \rangle.$$

Since f_n is supported on $[0, na)$, $T_t f_n$ is supported on $[t, na+t)$ and $T_s(f_n)$ is supported on $[s, na+s)$. We compute, using the change of variables $y = x - s$ at the second step, and with the last step justified afterwards,

$$\begin{aligned}
\langle T_t(f_n), \, T_s(f_n) \rangle &= \int_s^{na+t} f_n(x-t) \overline{f_n(x-s)} \, dm(x) \\
&= \int_0^a f_n(y) \overline{f_n(y-a)} \, dm(y) + \int_a^{na} f_n(y) \overline{f_n(y-a)} \, dm(y) + \int_{na}^{(n+1)a} f_n(y) \overline{f_n(y-a)} \, dm(y) \\
&= -(n-1)a.
\end{aligned}$$

The last step follows from the relations $f_n(y)\overline{f_n(y-a)} = 0$ for $y \in [0, a)$ (because the second factor is zero), $f_n(y)\overline{f_n(y-a)} = 0$ for $y \in [na, \, (n+1)a)$ (because the first factor is zero), and $f_n(y)\overline{f_n(y-a)} = -1$ for $y \in [a, na)$ (because at every point in $[a, na)$, one factor is 1 and the other is -1). It follows that

$$\|(T_t - T_s) f_n\|^2 = (4n - 2)a.$$

This together with $\|f_n\|^2 = na$ proves that

$$\|T_t - T_s\| \geq \sqrt{4 - \frac{2}{n}}.$$

By letting $n \to \infty$, we see that $\|T_t - T_s\| \geq 2$. □

Note that this proves $\|T_t - T_s\| = 2$ for $t \neq s$, using $\|T_t\| = 1$ and the triangle inequality.

The only essential difference in the alternate solution given next is the choice of function, namely a complex exponential instead of an alternating sum of characteristic functions of intervals. The solution is organized somewhat differently: no sequences, no mention of scalar products, and we use functions of norm 1. Any or all of these changes could have been used in the first solution, and any or all of the corresponding parts of the first solution could have been used in the alternate solution.

Alternate solution. By exchanging s and t, we can assume $s < t$. Let $\epsilon > 0$; we find $f \in L^2(\mathbb{R})$ such that $\|(T_t - T_s) f\| > 2 - \epsilon$. Without loss of generality, $\epsilon < 1$. Define $M = (t-s)/\epsilon$. Define $f \in L^2(\mathbb{R})$ by

$$f(x) = \begin{cases} (2M)^{-1/2} \exp(\pi i x/(t-s)) & x \in [-M, \, M] \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f\| = 1$. For $x \in \mathbb{R}$ we have

$$(T_t f)(x) = \begin{cases} (2M)^{-1/2} \exp\bigl(\pi i(x-t)/(t-s)\bigr) & x \in [-M+t,\, M+t] \\ 0 & \text{otherwise} \end{cases}$$

and

$$(T_s f)(x) = \begin{cases} (2M)^{-1/2} \exp\bigl(\pi i(x-s)/(t-s)\bigr) & x \in [-M+s,\, M+s] \\ 0 & \text{otherwise.} \end{cases}$$

So, for x in the set

$$E = [-M+t,\, M+t] \cap [-M+s,\, M+s] = [-M+t,\, M+s],$$

we have

$$\begin{aligned} |(T_t f)(x) - (T_s f)(x)| &= \left(\frac{1}{\sqrt{2M}}\right) \bigl|\exp\bigl(\pi i(x-t)/(t-s)\bigr) - \exp\bigl(\pi i(x-s)/(t-s)\bigr)\bigr| \\ &= \left(\frac{1}{\sqrt{2M}}\right) \bigl|\exp(-\pi i)\exp\bigl(\pi i(x-s)/(t-s)\bigr) - \exp\bigl(\pi i(x-s)/(t-s)\bigr)\bigr| \\ &= \left(\frac{1}{\sqrt{2M}}\right) \bigl|-2\exp\bigl(\pi i(x-s)/(t-s)\bigr)\bigr| = \frac{2}{\sqrt{2M}}. \end{aligned}$$

Since E has measure $2M - (t-s)$, we get, using $\epsilon < 1$ at the fourth step,

$$\|(T_t - T_s)f\| \geq (2M - (t-s)) \left(\frac{2}{\sqrt{2M}}\right)^2 = 4\left(1 - \frac{t-s}{2M}\right) = 4 - 2\epsilon > 4 - 4\epsilon + \epsilon^2 = (2-\epsilon)^2.$$

So $\|(T_t - T_s)f\| > 2 - \epsilon$, as desired. □

7. Suppose $f\colon D \to \mathbb{C}$ is a holomorphic function on the open unit disk $D = \{z \in \mathbb{C}\colon |z| < 1\}$. If f is injective on $D \setminus \{0\}$, prove that f is injective on D.

Solution. Suppose f is not injective on D. Then there exists $z_0 \in D \setminus \{0\}$ such that $f(z_0) = f(0)$. Choose disjoint open subsets $V, W \subset D$ such that $0 \in V$ and $z_0 \in W$. Clearly f is not constant, so f is an open mapping, and $f(V)$ and $f(W)$ are open. Now $f(0) \in f(V) \cap f(W)$, and $\{f(0)\}$ is not open, so there is some $w \in f(V) \cap f(W)$ with $w \neq f(0)$. There are $a \in V$ and $b \in W$ such that $f(a) = f(b) = w$. Clearly $a \neq b$ and neither a nor b is zero. So f is not injective on $D \setminus \{0\}$. □

Alternate solution. If f' is constant, then either f is constant or there are $a, b \in \mathbb{C}$ such that $f(z) = az + b$ for all $z \in D$. In either case, the statement of the problem is clear. So we can assume f' is not constant. Replacing f by $f - f(0)$, we can further assume that $f(0) = 0$.

Suppose f is not injective on D. Then there exists $z_0 \in D \setminus \{0\}$ such that $f(z_0) = 0$. Choose r such that $|z_0| < r < 1$. Set $C = \frac{1}{2}\inf_{|z|=r} |f(z)|$. Since f is injective on $D \setminus \{0\}$ and $f(z_0) = 0$, it follows that $f(z) \neq 0$ when $|z| = r$, so $C > 0$. Choose $\delta > 0$ such that $\delta < r$ and $|z| < \delta$ implies $|f(z)| < C$. Since f' is not constant, its zeros are isolated, and there is $w \in D$ such that $0 < |w| < \delta$ and $f'(w) \neq 0$. Therefore the function $g(z) = f(z) - f(w)$ has a simple zero at w. Since $w \neq 0$, the injectivity hypothesis implies that g has no other zeros in D.

When $|z| = r$, we have

$$|g(z) - f(z)| = |f(w)| < C < |f(z)|.$$

So Rouché's Theorem implies that f and g have the same number of zeros in $\{z \in \mathbb{C}\colon |z| < r\}$, counting multiplicity. But we saw that g has only one zero, which has multiplicity 1, while f has at least two zeros. This contradiction shows that f is injective on D. □

Second alternate solution (sketch). In the alternate solution, instead of using f' not constant to choose w, choose w arbitrarily and use the fact that if $f - f(w)$ has a zero at w of multiplicity more than 1, then f is not injective on any neighborhood of w. □

8. Suppose f is an entire function such that $|f(z)| \leq 1 + \sqrt{|z|}$ for all $z \in \mathbb{C}$. Prove that f is constant.

Solution. Let $n \in \mathbb{N}$. Using Cauchy's Estimate at the first step, for every $R > 0$ we have

$$|f^{(n)}(0)| \leq n! R^{-n} \sup_{|z|<R} |f(z)| \leq n! R^{-n}\big(1 + \sqrt{R}\big).$$

Since $\lim_{R\to\infty} n! R^{-n}\big(1+\sqrt{R}\big) = 0$, it follows that $f^n(0) = 0$. Since this is true for all $n \in \mathbb{N}$, f is constant. □

Alternate solution. Set $g(z) = f\big(\frac{1}{z}\big)$ for $z \in \mathbb{C} \setminus \{0\}$. Then

$$|g(z)| \leq 1 + \frac{1}{\sqrt{|z|}}$$

for $z \in \mathbb{C} \setminus \{0\}$. Therefore

$$|zg(z)| \leq z + \sqrt{|z|}$$

for $z \in \mathbb{C} \setminus \{0\}$. In particular, $z \mapsto zg(z)$ is bounded on $\big\{z \in \mathbb{C} \colon 0 < |z| < 1\big\}$, and therefore has a removable singularity at 0. So there is a holomorphic function h on $\mathbb{C}$ such that $h(z) = zg(z)$ for all $z \in \mathbb{C} \setminus \{0\}$. Clearly $h(0) = 0$. Therefore there is a holomorphic function k on $\mathbb{C}$ such that $h(z) = zk(z)$ for all $z \in \mathbb{C} \setminus \{0\}$. Then

$$\lim_{z\to\infty} f(z) = \lim_{z\to 0} g(z) = k(0).$$

In particular, $\lim_{z\to\infty} f(z)$ exists, so f is bounded. Liouville's Theorem now tells us that f is constant. □

9. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos(\theta)}$.

Solution. We rewrite the integral as a contour integral over the positively oriented unit circle C. Set

$$f(z) = \frac{i}{z^2 - 3z + 1}$$

for all $z \in \mathbb{C}$ for which $z^2 - 3z + 1 \neq 0$. Parametrize C via $\gamma(t) = \exp(i\theta)$ for $\theta \in [0, 2\pi]$. Then

$$\int_C f(z)\, dz = \int_0^{2\pi} \frac{i \cdot i \exp(i\theta)\, d\theta}{\exp(i\theta)^2 - 3\exp(i\theta) + 1} = \int_0^{2\pi} \frac{(-1)\, d\theta}{\exp(i\theta) - 3 + \exp(-i\theta)} = \int_0^{2\pi} \frac{d\theta}{3 - 2\cos(\theta)}.$$

The function f has simple poles at $\big(3 \pm \sqrt{5}\big)/2$. Only $z_0 = \big(3 - \sqrt{5}\big)/2$ is inside the unit circle C.

We calculate the residue of f at z_0. Write

$$f(z) = \frac{i}{\Big(z - \frac{3+\sqrt{5}}{2}\Big)\Big(z - \frac{3-\sqrt{5}}{2}\Big)} = \left(\frac{i}{z - \frac{3+\sqrt{5}}{2}}\right)\left(\frac{1}{z - \frac{3-\sqrt{5}}{2}}\right)$$

and evaluate the first factor at z_0, getting

$$\operatorname{Res}(f;\, z_0) = \frac{i}{\frac{3-\sqrt{5}}{2} - \frac{3+\sqrt{5}}{2}} = -\frac{i}{\sqrt{5}}.$$

By the Residue Theorem,

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos(\theta)} = \int_C f(z)\, dz = 2\pi i\left(-\frac{i}{\sqrt{5}}\right) = \frac{2\pi}{\sqrt{5}}.$$

□