Analysis Qualifying Exam - SOLUTIONS, Fall 2019

1. Let (X, μ) be a positive measure space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of measure sets such that

$$\mu(A_n) < n^{-20/19} \qquad \text{for all } n \in \mathbb{N}.$$

Show that

$$\mu\left(\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} A_j\right) = 0.$$

Solution: For $i \in \mathbb{N}$ define the set

$$B_i = \bigcup_{j=i}^{\infty} A_j.$$

These sets are nested $B_i \supset B_{i+1}$ and

$$\mu(B_1) \le \sum_{n=1}^{\infty} \mu(A_n) \le \sum_{n=1}^{\infty} \frac{1}{n^{20/19}} < \infty.$$

The above series converges since it is harmonic series $\sum 1/n^p$ with $p = 20/19 > 1$. By the monotonicity property of measure we have

$$\mu(B_n) \to \mu\left(\bigcap_{i=1}^{\infty} B_i\right) \qquad \text{as } n \to \infty.$$

Since the tail of convergent series converges to zero we conclude that

$$\mu(B_n) \le \sum_{i=n}^{\infty} \mu(A_i) \le \sum_{i=n}^{\infty} \frac{1}{i^{20/19}} \to 0 \qquad \text{as } n \to \infty.$$

□

2. Let (X, μ) be a positive measure space such that $\mu(X) = 1$. Let f be a complex measurable function on X. Define $\varphi : (0, \infty) \to [0, \infty]$ by

$$\varphi(p) = \left(\int_X |f(x)|^p d\mu(x)\right)^{1/p} \qquad p \in (0, \infty).$$

Note that we allow φ to take the value ∞. Prove that φ is a nondecreasing function.

Solution: Take any $0 < p_1 < p_2 < \infty$. Let $p > 1$ be such that $pp_1 = p_2$ and let p' be the conjugate exponent $1/p + 1/p' = 1$. Suppose first that $\varphi(p_1) < \infty$. By Hölder's inequality

$$\varphi(p_1)^{p_1} \le \int_X |f(x)|^{p_1} d\mu(x) \le \left(\int_X |f(x)|^{pp_1} d\mu(x)\right)^{1/p} \left(\int_X d\mu(x)\right)^{1/p'} = \varphi(p_2)^{p_1}.$$

Next suppose that $\varphi(p_1) = \infty$. Let

$$X' = \{x \in X : |f(x)| \ge 1\}.$$

Since $\mu(X) = 1$ we have

$$\infty = \int_{X'} |f(x)|^{p_1} d\mu(x) + \int_{X\setminus X'} |f(x)|^{p_1} d\mu(x) \le \int_{X'} |f(x)|^{p_1} d\mu(x) + 1.$$

Thus,

$$\infty = \int_{X'} |f(x)|^{p_1} d\mu(x) \le \int_{X'} |f(x)|^{p_2} d\mu(x).$$

Hence, φ is a nondecreasing function. $\square$

3. Let (X, μ) be a positive measure space and let $p \in [1, \infty)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex measurable functions on X such that $|f_1| \ge |f_2| \ge |f_3| \ge \ldots$ Suppose that $f_{2020} \in L^p(\mu)$ and $f(x) = \lim_{n\to\infty} f_n(x)$ exists for a.e. $x \in X$. Prove that (f_n) converges to f in $L^p(\mu)$ norm.

Solution: Let $g_n = f - f_n$. For $n \ge 2020$

$$|g_n(x)| \le |f(x)| + |f_n(x)| \le 2|f_{2020}(x)|.$$

Since $g_n(x) \to 0$ as $n \to \infty$ for a.e. x and

$$|f_{2020}|^p \in L^1(\mu),$$

the Lebesgue dominated convergence theorem yields

$$\int_X |g_n(x)|^p d\mu(x) \to 0 \qquad \text{as } n \to \infty.$$

$\square$

4. Let L be a continuous linear functional on a Hilbert space $\mathcal{H}$. Prove that $(\ker L)^\perp$ is a vector space of dimension at most 1.

Solution: By the Riesz representation theorem there exists a unique vector $y \in \mathcal{H}$ such that

$$Lx = \langle x, y\rangle \qquad \text{for all } x \in \mathcal{H}.$$

Let V be the linear span of y, which is 1 dimensional subspace of $\mathcal{H}$ if $y \ne 0$. Naturally, $V = \{0\}$ if $y = 0$. Hence,

$$\ker L = \{x \in \mathcal{H} : \langle x, y\rangle = 0\} = V^\perp.$$

Since $\mathcal{H} = V \oplus V^\perp$ we have $(V^\perp)^\perp = V$. Thus,

$$(\ker L)^\perp = V.$$

$\square$

5. Suppose that $(x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$, $1 \le p < \infty$. Define a set

$$S = \{(y_n)_{n\in\mathbb{N}} : |y_n| \le |x_n| \quad \text{for all } n \in \mathbb{N}\}.$$

Prove that S is a compact subset of $\ell^p(\mathbb{N})$.

Solution: Take any sequence $(\boldsymbol{x}^m)_{m\in\mathbb{N}}$ of points in S. We write $\boldsymbol{x}^m = (x_n^m)_{n\in\mathbb{N}}$. Since $(x_1^m)_{m\in\mathbb{N}}$ is a bounded sequence we can find an infinite subset $I_1 \subset \mathbb{N}$ such that the limit below exists

$$z_1 := \lim_{I_1 \ni m\to\infty} x_1^m.$$

Having defined infinite sets $I_1 \supset \ldots \supset I_n$, we define infinite set I_{n+1} as follows. Since $(x_{n+1}^m)_{m\in I_n}$ is a bounded sequence we can find an infinite subset $I_{n+1} \subset I_n$ such that the limit below exists

$$z_{n+1} := \lim_{I_{n+1} \ni m\to\infty} x_{n+1}^m.$$

Finally, we use the diagonal argument to define a set $I \subset \mathbb{N}$ which contains n'th largest element from each of the sets I_n. Consequently, for any $n \in \mathbb{N}$ we have

$$z_n = \lim_{I \ni m \to \infty} x_n^m.$$

Consequently, the subsequence $(\boldsymbol{x}^m)_{m \in I}$ converges pointwise to $\boldsymbol{z} = (z_n)_{n \in \mathbb{N}}$ and

$$|x_n^m|^p \le |y_n|^p \qquad \text{and} \qquad \sum_{n=1}^{\infty} |y_n|^p < \infty.$$

The Lebesgue dominated convergence theorem for the counting measure on $\mathbb{N}$ yields

$$\sum_{n=1}^{\infty} |x_n^m - z_n|^p \to 0 \qquad \text{as } I \ni m \to \infty.$$

$\square$

6. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions in $L^p(\mathbb{R})$, $1 < p < \infty$. Suppose that for any $g \in L^q(\mathbb{R})$, $1/p + 1/q = 1$,

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}} |f_n(x) g(x)| dx < \infty.$$

Prove that

$$\sup_{n \in \mathbb{N}} ||f_n||_p < \infty.$$

Solution: By the duality $(L^q)^* = L^p$ we define bounded linear functionals L_n on $L^q(\mathbb{R})$ by

$$L_n g = \int_{\mathbb{R}} f_n(x) g(x) dx \qquad g \in L^q(\mathbb{R}),\ n \in \mathbb{N}.$$

By our hypothesis for any $g \in L^q(\mathbb{R})$ we have

$$\sup_{n \in \mathbb{N}} |L_n(g)| < \infty.$$

By the Banach-Steinhaus principle of uniform boundedness the operator norms $||L_n||$ are uniformly bounded. Finally, we invoke again the duality $||L_n|| = ||f_n||_p$. $\square$

7. Prove that that finite linear combinations of $\{e^{-|x-a|} : a \in \mathbb{R}\}$ are dense in $L^2(\mathbb{R})$. Hint: apply Wiener's theorem about translation invariant subspaces of $L^2(\mathbb{R})$.

Solution: Let V be the closed linear span of $\{e^{-|x-a|} : a \in \mathbb{R}\}$ in $L^2(\mathbb{R})$ norm. The subspace $V \subset L^2(\mathbb{R})$ is translation invariant. That is if $f \in V$, then any translate $f(\cdot - k)$, $k \in \mathbb{R}$, also belongs to V. By Wiener's theorem there exists a measurable subset $E \subset \mathbb{R}$ such that

$$V = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for a.e. } \xi \in \mathbb{R} \setminus E\}.$$

Let $f(x) = e^{-|x|}$. The Fourier transform of f is

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-|x|} e^{-ix\xi} dx = \int_{-\infty}^{0} e^{x(1-i\xi)} dx + \int_{0}^{\infty} e^{-x(1+i\xi)} dx = \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2}.$$

Since $\hat{f}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, by Wiener's theorem we have $E = \mathbb{R}$ (modulo null sets). Thus, $V = L^2(\mathbb{R})$. $\square$

8. Suppose that an entire function f satisfies

$$|f(z)| \le |z|^{19/20} \qquad \text{for } |z| > 2019.$$

Prove that f is constant.

Solution: For $r > 0$, let γ_r be the positively oriented circle $|z| = r$. For any z_0 such that $|z_0| < r$, Cauchy's formula states that

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z - z_0} dz.$$

Differentiating this over z_0 we obtain Cauchy's formula for derivative of f

$$f'(z_0) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{(z - z_0)^2} dz.$$

Hence,

$$|f'(z_0)| \le \frac{1}{2\pi}\int_{\gamma_r} \frac{|f(z)|}{|z - z_0|^2} d|z| \le \frac{1}{2\pi} 2\pi r \frac{|r|^{19/20}}{(r - |z_0|)^2} = \frac{r^{39/40}}{r^2 - 2|z_0|r + |z_0|^2} \to 0 \qquad \text{as } r \to \infty.$$

Hence, $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$ and f is constant. □

9. Let Ω be a domain containing a closed unit disk. Suppose that $f \in H(\Omega)$ is such that $f(1/2) = 2019$ and

$$|f(z)| > 2020 \qquad \text{for all } |z| = 1.$$

Show that f has at least one zero in the open unit disk.

Solution: Let $g(z) = f(z) - 2019$. Note that for $|z| = 1$

$$|f(z) - g(z)| = 2019 < 2020 < |f(z)|.$$

By Rouche's theorem f and g have the same number of zeros inside the open unit disk. Since $g(1/2) = 0$, f has at least one zero. □