1. Let $(X, \mu)$ be a positive measure space. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of measure sets such that
\[ \mu(A_n) < n^{-20/19} \quad \text{for all } n \in \mathbb{N}. \]
Show that
\[ \mu \left( \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j \right) = 0. \]

2. Let $(X, \mu)$ be a positive measure space such that $\mu(X) = 1$. Let $f$ be a complex measurable function on $X$. Define $\varphi : (0, \infty) \to [0, \infty]$ by
\[ \varphi(p) = \left( \int_X |f(x)|^p d\mu(x) \right)^{1/p} \quad p \in (0, \infty). \]
Note that we allow $\varphi$ to take the value $\infty$. Prove that $\varphi$ is a nondecreasing function.

3. Let $(X, \mu)$ be a positive measure space and let $p \in [1, \infty)$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of complex measurable functions on $X$ such that $|f_1| \geq |f_2| \geq |f_3| \geq \ldots$. Suppose that $f_{2020} \in L^p(\mu)$ and $f(x) = \lim_{n \to \infty} f_n(x)$ exists for a.e. $x \in X$. Prove that $(f_n)$ converges to $f$ in $L^p(\mu)$ norm.

4. Let $L$ be a continuous linear functional on a Hilbert space $H$. Prove that $(\ker L)^\perp$ is a vector space of dimension at most 1.

5. Suppose that $(x_n)_{n \in \mathbb{N}} \in \ell^p(\mathbb{N})$, $1 \leq p < \infty$. Define a set
\[ S = \{(y_n)_{n \in \mathbb{N}} : |y_n| \leq |x_n| \quad \text{for all } n \in \mathbb{N}\}. \]
Prove that $S$ is a compact subset of $\ell^p(\mathbb{N})$.

6. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions in $L^p(\mathbb{R})$, $1 < p < \infty$. Suppose that for any $g \in L^q(\mathbb{R})$, $1/p + 1/q = 1$,
\[ \sup_{n \in \mathbb{N}} \int_{\mathbb{R}} |f_n(x)|^{|p|} dx < \infty. \]
Prove that
\[ \sup_{n \in \mathbb{N}} ||f_n||_p < \infty. \]

7. Prove that that finite linear combinations of $\{e^{-|x-a|} : a \in \mathbb{R}\}$ are dense in $L^2(\mathbb{R})$. Hint: apply Wiener’s theorem about translation invariant subspaces of $L^2(\mathbb{R})$.

8. Suppose that an entire function $f$ satisfies
\[ |f(z)| \leq |z|^{19/20} \quad \text{for } |z| > 2019. \]
Prove that $f$ is constant.

9. Let $\Omega$ be a domain containing a closed unit disk. Suppose that $f \in H(\Omega)$ is such that $f(1/2) = 2019$ and
\[ |f(z)| > 2020 \quad \text{for all } |z| = 1. \]
Show that $f$ has at least one zero in the open unit disk.