Analysis Qualifying Exam, Fall 2019

1. Let (X,μ) be a positive measure space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of measure sets such that

$$\mu(A_n) < n^{-20/19} \qquad \text{for all } n \in \mathbb{N}.$$

Show that

$$\mu\left(\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} A_j\right) = 0.$$

2. Let (X,μ) be a positive measure space such that $\mu(X) = 1$. Let f be a complex measurable function on X. Define $\varphi : (0,\infty) \to [0,\infty]$ by

$$\varphi(p) = \left(\int_X |f(x)|^p d\mu(x)\right)^{1/p} \qquad p \in (0,\infty).$$

Note that we allow φ to take the value ∞. Prove that φ is a nondecreasing function.

3. Let (X,μ) be a positive measure space and let $p \in [1,\infty)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex measurable functions on X such that $|f_1| \geq |f_2| \geq |f_3| \geq \ldots$ Suppose that $f_{2020} \in L^p(\mu)$ and $f(x) = \lim_{n\to\infty} f_n(x)$ exists for a.e. $x \in X$. Prove that (f_n) converges to f in $L^p(\mu)$ norm.

4. Let L be a continuous linear functional on a Hilbert space $\mathcal{H}$. Prove that $(\ker L)^{\perp}$ is a vector space of dimension at most 1.

5. Suppose that $(x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$, $1 \leq p < \infty$. Define a set

$$S = \{(y_n)_{n\in\mathbb{N}} : |y_n| \leq |x_n| \quad \text{for all } n \in \mathbb{N}\}.$$

Prove that S is a compact subset of $\ell^p(\mathbb{N})$.

6. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^p(\mathbb{R})$, $1 < p < \infty$. Suppose that for any $g \in L^q(\mathbb{R})$, $1/p + 1/q = 1$,

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} |f_n(x)g(x)|dx < \infty.$$

Prove that

$$\sup_{n\in\mathbb{N}} ||f_n||_p < \infty.$$

7. Prove that that finite linear combinations of $\{e^{-|x-a|} : a \in \mathbb{R}\}$ are dense in $L^2(\mathbb{R})$. Hint: apply Wiener's theorem about translation invariant subspaces of $L^2(\mathbb{R})$.

8. Suppose that an entire function f satisfies

$$|f(z)| \leq |z|^{19/20} \qquad \text{for } |z| > 2019.$$

Prove that f is constant.

9. Let Ω be a domain containing a closed unit disk. Suppose that $f \in H(\Omega)$ is such that $f(1/2) = 2019$ and

$$|f(z)| > 2020 \qquad \text{for all } |z| = 1.$$

Show that f has at least one zero in the open unit disk.