SOLUTIONS FOR ANALYSIS QUALIFYING EXAM, FALL 2018

Notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$. (That is, $0 \notin \mathbb{N}$.)

1. Let $(X, \mathfrak{M})$ be a measurable space with a σ-finite positive measure μ. Prove that there is a finite positive measure ν on $(X, \mathfrak{M})$ such that $\mu \ll \nu$ and $\nu \ll \mu$. (Recall that $\mu \ll \nu$ means that μ is absolutely continuous with respect to ν, that is, for any $E \in \mathfrak{M}$, $\nu(E) = 0$ implies $\mu(E) = 0$).

Solution. If μ is finite, take $\nu = \mu$. Hence we assume that $\mu(X) = \infty$. By the σ-finiteness assumption, we can write $X = \coprod_{n=1}^{\infty} F_n$ for measurable sets F_n such that $\mu(F_n) < \infty$ for each $n \in \mathbb{N}$. For any measurable set $A \in \mathfrak{M}$, define

$$\nu(A) = \sum_{n=1}^{\infty} \frac{\mu(A \cap F_n)}{2^n(1 + \mu(F_n))}.$$

One checks that ν is a measure by writing $\nu(A) = \int_A f \, d\mu$ with

$$f(x) = \frac{1}{2^n(1 + \mu(F_n))}$$

whenever $x \in F_n$. (The function f can also be written as

$$f = \sum_{n=1}^{\infty} \left(\frac{1}{2^n(1 + \mu(F_n))} \right) \chi_{F_n}.$$

Alternatively, show directly that ν is a measure: clearly $\nu(\varnothing) = 0$, and countable additivity of ν follows from countable additivity of μ.) Clearly $\nu(X) < 1$.

If $\mu(A) = 0$, then $\mu(A \cap F_n) = 0$ for each $n \in \mathbb{N}$, whence $\nu(A) = 0$. On the other hand, if $\nu(A) = 0$, then $\mu(A \cap F_n) = 0$ for each $n \in \mathbb{N}$. This implies that $\mu(A) = 0$ by countable additivity. $\square$

2. Let (X, μ) be a measure space with $\mu(X) < \infty$. For $f \in L^2(X, \mu)$, prove that $|f| \log(|f|)$ is in $L^1(X, \mu)$. (Take $y \log(y) = 0$ when $y = 0$.) If $(X, \mu) = (\mathbb{R}, m)$, is it still the case that $f \in L^2$ implies $|f| \log(|f|) \in L^1(X, \mu)$?

Solution. First consider the case $\mu(X) < \infty$. Since $y \mapsto y \log(y)$ is continuous on $(0, 1]$ and $\lim_{y \to 0^+} y \log(y)$ exists (it is zero), there is a constant M such that $|y \log(y)| \leq M$ for all $y \in [0, 1]$. (In fact, methods of elementary calculus show that for all $y \in (0, 1]$, we have $0 \geq y \log(y) \geq -\frac{1}{e}$, so we can take $M = \frac{1}{e}$. But we don't need the exact bound.)

Set

$$E = \big\{ x \in X \colon |f(x)| \geq 1 \big\}.$$

For $x \in E$, $0 \leq \log\big(|f(x)|\big) < |f(x)|$. Hence $0 \leq |f(x)| \log\big(|f(x)|\big) < |f(x)|^2$. Therefore, using $f \in L^2(X, \mu)$ at the last step,

$$\begin{aligned}
\int_X \big| |f(x)| \log\big(|f(x)|\big) \big| \, d\mu &= \int_E |f(x)| \log\big(|f(x)|\big) \, d\mu + \int_{X \setminus E} \big| |f(x)| \log\big(|f(x)|\big) \big| \, d\mu \\
&\leq \int_E |f|^2 \, d\mu + \int_{X \setminus E} M \, d\mu \\
&\leq \int_X |f|^2 \, d\mu + M\mu(X \setminus E) \leq \int_X |f|^2 \, d\mu + M\mu(X) < \infty.
\end{aligned}$$

If $(X, \mu) = (\mathbb{R}, m)$, the statement fails. Here is a counterexample. (A counterexample, with proof, must be present in a correct solution.) Set

$$f(x) = \begin{cases} x^{-3/4} & x \geq 1 \\ 0 & x < 1. \end{cases}$$

Then

$$\int_{\mathbb{R}} |f|^2 \, dm = \int_1^{\infty} x^{-3/2} \, dx = 2,$$

but

$$\int_{\mathbb{R}} \big| |f| \log(|f|) \big| \, dm = -\int_1^\infty x^{-3/4} \log(x^{-3/4}) \, dx = \frac{3}{4} \int_1^\infty x^{-3/4} \log(x) \, dx > \frac{3}{4} \int_e^\infty x^{-3/4} \, dx = \infty.$$

This completes the solution. □

Alternate solution to the second part (sketch). For any $\alpha \in \left(\frac{1}{2}, 1\right]$, the function

$$f(x) = \begin{cases} x^{-\alpha} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

is a counterexample. There are many others. □

3. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function.

(a) Prove that $\lim_{n\to\infty} \int_0^1 x^n f(x) \, dx = 0$. (3 points).

(b) Prove that $\lim_{n\to\infty} \int_0^1 (n+1) x^n f(x) \, dx = f(1)$. (7 points)

*Solution to **3**.(a).* For $n \in \mathbb{N}$, set $g_n(x) = x^n f(x)$ for all $x \in [0,1]$. Then $\lim_{n\to\infty} g_n(x) = 0$ for almost all $x \in [0,1]$ (in fact, for all $x \in [0,1]$ except possibly $x = 1$), $|f|$ bounded and hence is integrable on $[0,1]$, and $|g_n(x)| \leq |f(x)|$ for all $x \in [0,1]$ and all $n \in \mathbb{N}$. The Lebesgue Dominated Convergence Theorem therefore implies that

$$\lim_{n\to\infty} \int_0^1 x^n f(x) \, dx = \int_0^1 \left(\lim_{n\to\infty} g_n(x) \right) dx = 0.$$

This completes the solution. □

*Alternate solution to **3**.(a).* Let $\epsilon > 0$. We find $N \in \mathbb{N}$ such that $n \geq N$ implies $\left| \int_0^1 x^n f(x) \, dx \right| < \epsilon$. Set

$$M = \sup_{x \in [0,1]} |f(x)| \qquad \text{and} \qquad \alpha = 1 - \frac{\epsilon}{2M+1}.$$

Without loss of generality, ϵ is small enough that $\alpha > 0$. Choose $N \in \mathbb{N}$ so large that

$$\alpha^N < \frac{\epsilon}{2M+1}.$$

Let $n \in \mathbb{N}$ satisfy $n \geq N$. Then

$$\left| \int_0^1 x^n f(x) \, dx \right| \leq \int_0^1 x^n |f(x)| \, dx = \int_0^\alpha x^n |f(x)| \, dx + \int_\alpha^1 x^n |f(x)| \, dx \leq \int_0^\alpha \alpha^n M \, dx + \int_\alpha^1 M \, dx$$
$$= \alpha \cdot M\alpha^n + M(1-\alpha) < M\alpha^N + M(1-\alpha) < M\left(\frac{\epsilon}{2M+1}\right) + M\left(\frac{\epsilon}{2M+1}\right) < \epsilon.$$

This completes the solution. □

*Solution to **3**.(b).* Using the change of variables $y = x^{n+1}$, so $x = y^{1/(n+1)}$ and $dy = (n+1)x^n \, dx$, we get

$$\int_0^1 (n+1) x^n f(x) \, dx = \int_0^1 f\left(y^{1/(n+1)}\right) dy.$$

Set $M = \sup_{x \in [0,1]} |f(x)|$. For $n \in \mathbb{N}$, set $g_n(x) = f\left(y^{1/(n+1)}\right)$ for all $y \in [0,1]$. Then $\lim_{n\to\infty} g_n(y) = f(1)$ for all $y \in [0,1]$ except $y = 0$, the constant function M is integrable on $[0,1]$, and $|g_n(y)| \leq M$ for all $y \in [0,1]$ and all $n \in \mathbb{N}$. The Lebesgue Dominated Convergence Theorem therefore implies that

$$\lim_{n\to\infty} \int_0^1 g_n(x) \, dx = \int_0^1 f(1) \, dx = f(1).$$

This completes the solution. □

We can also prove this by direct estimates.

Alternate solution to 3.(b). Let $\epsilon > 0$. Set $M = \sup_{x\in[0,1]} |f(x)|$. Choose $\delta \in (0,1)$ sufficiently small that $|f(x) - f(1)| < \frac{\epsilon}{3}$ for $x \in [1-\delta,\, 1]$. Choose $N \in \mathbb{N}$ so large that

$$(1-\delta)^{N+1} < \frac{\epsilon}{3M+1}.$$

Let $n \in \mathbb{N}$ satisfy $n \geq N$. Then

$$\begin{aligned}\int_0^1 (n+1)x^n f(x)\,dx &= \int_{1-\delta}^1 (n+1)x^n f(x)\,dx + \int_0^{1-\delta} (n+1)x^n f(x)\,dx \\ &= \int_{1-\delta}^1 (n+1)x^n f(1)\,dx + \int_{1-\delta}^1 (n+1)x^n [f(x)-f(1)]\,dx + \int_0^{1-\delta} (n+1)x^n f(x)\,dx.\end{aligned}$$

Now

$$\left| \int_{1-\delta}^1 (n+1)x^n f(1)\,dx - f(1) \right| = \left| f(1)[1-(1-\delta)^{n+1}] - f(1) \right| = (1-\delta)^{n+1}|f(1)| < \frac{\epsilon}{3},$$

$$\left| \int_{1-\delta}^1 (n+1)x^n [f(x)-f(1)]\,dx \right| \leq \frac{\epsilon}{3}\int_0^1 (n+1)x^n\,dx \leq \frac{\epsilon}{3},$$

and

$$\left| \int_0^{1-\delta} (n+1)x^n f(x)\,dx \right| < M(1-\delta)^{n+1} < \frac{\epsilon}{3}.$$

Putting the last three inequalities together gives

$$\left| \int_0^1 (n+1)x^n f(x)\,dx - f(1) \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This completes the solution. □

4. Let $C([0,1])$ be the space of continuous functions on $[0,1]$. Prove that

$$\|f\| = \left(\int_{[0,1]} |f|^2\,dm \right)^{1/2}$$

defines a norm on $C([0,1])$. Is $C([0,1])$ a Banach space with respect to this norm? Justify your answer.

We give a direct proof first.

Solution. We check that the formula given defines a norm. It is obvious that $\|f\| \geq 0$ for all $f \in C([0,1])$ and that $\|0\| = 0$. It is clear that $\|\alpha f\| = |\alpha|\|f\|$ for all $f \in C([0,1])$ and all $\alpha \in \mathbb{C}$. Next, suppose that $f \in C([0,1])$ and $\|f\| = 0$. Then $f = 0$ almost everywhere with respect to m. For every nonempty open set $U \subset [0,1]$, we have $m(U) > 0$. So $W = \{x \in [0,1\colon f(x) \neq 0\}$ contains no nonempty open set. Since f is continuous, $W = \varnothing$. Thus f is the zero element of $C([0,1])$.

Finally, let $f, g \in C([0,1])$; we claim that $\|f+g\| \leq \|f\| + \|g\|$. We may assume $\|f+g\| \neq 0$. We have, using Hölder's inequality at the fourth step,

$$\begin{aligned}\|f+g\|^2 &= \int_{[0,1]} |f+g|^2\,dm \leq \int_{[0,1]} |f+g|(|f|+|g|)\,dm = \int_{[0,1]} |f+g||f|\,dm + \int_{[0,1]} |f+g||g|\,dm \\ &\leq \left(\int_{[0,1]} |f+g|^2\,dm\right)^{1/2}\left(\int_{[0,1]} |f|^2\,dm\right)^{1/2} + \left(\int_{[0,1]} |f+g|^2\,dm\right)^{1/2}\left(\int_{[0,1]} |g|^2\,dm\right)^{1/2} \\ &= \|f+g\|\big(\|f\| + \|g\|\big).\end{aligned}$$

The claim follows by dividing by $\|f+g\|$.

However, $C([0,1])$ with this norm is not a Banach space, since $C([0,1])$ is not complete with respect this norm. There are many examples one can construct. For example, for $n \in \mathbb{N}$ define $f_n \in C([0,1)$ to be the piecewise linear function

$$f_n(x) = \begin{cases} 1 & x \in \left[0,\, \frac{1}{2} - \frac{1}{2n}\right] \\ \frac{1}{2} - n\left(x - \frac{1}{2}\right) & x \in \left(\frac{1}{2} - \frac{1}{2n},\, \frac{1}{2} + \frac{1}{2n}\right) \\ 0 & x \in \left[\frac{1}{2} - \frac{1}{2n},\, 1\right]. \end{cases}$$

We claim that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ with $N > 4/\epsilon^2$. Let $m, n \in \mathbb{N}$ satisfy $m, n \geq N$. Then $f_m(x) = f_n(x) = 0$ for $x \in [0,1] \setminus \left[\frac{1}{2} - \frac{1}{2N}, \frac{1}{2} + \frac{1}{2N}\right]$ and $|f_m(x) - f_n(x)| \leq f_m(x) + f_n(x) \leq 2$ for $x \in \left[\frac{1}{2} - \frac{1}{2N}, \frac{1}{2} + \frac{1}{2N}\right]$. Therefore

$$\|f_m - f_n\|^2 \leq 4m\left(\left[\frac{1}{2} - \frac{1}{2N}, \frac{1}{2} + \frac{1}{2N}\right]\right) = \frac{4}{N} < \epsilon^2.$$

The claim is proved.

We finish the proof by showing that $\lim_{n\to\infty} f_n$ does not exist. Suppose that $f \in C([0,1])$ and $\lim_{n\to\infty} \|f_n - f\| = 0$. Since f_n is real for all $n \in \mathbb{N}$, it is easy to see that $\|f_n - \mathrm{Re}(f)\| \leq \|f_n - f\|$. Since limits are unique if they exist, f must be real.

Choose $\delta > 0$ such that $\left|x - \frac{1}{2}\right| < \delta$ implies $\left|f(x) - f\left(\frac{1}{2}\right)\right| < \frac{1}{4}$. First suppose that $f\left(\frac{1}{2}\right) \geq \frac{1}{2}$. For every $n > 2/\delta$, if $x \in \left(\frac{1}{2} + \frac{\delta}{4}, \frac{1}{2} + \frac{\delta}{2}\right]$, then $f(x) > \frac{1}{4}$ while $f_n(x) = 0$. So

$$\|f_n - f\| \geq \left(\int_{1/2+\delta/4}^{1/2+\delta/2} |f_n - f|^2 \, dm\right)^{1/2} \geq \left[\left(\frac{\delta}{4}\right)\left(\frac{1}{4}\right)^2\right]^{1/2} = \frac{\sqrt{\delta}}{8}.$$

This contradicts $\lim_{n\to\infty} \|f_n - f\| = 0$. Otherwise, $f\left(\frac{1}{2}\right) < \frac{1}{2}$. Then a similar argument on $\left[\frac{1}{2} - \frac{\delta}{2}, \frac{1}{2} - \frac{\delta}{4}\right]$ shows $\|f_n - f\| > \sqrt{\delta}/8$ for $n > 2/\delta$. Again, we have contradicted $\lim_{n\to\infty} \|f_n - f\| = 0$. This completes the solution. □

It is not enough to just say $f_n \to \chi_{[0,\frac{1}{2}]}$ in the norm given, and that $\chi_{[0,\frac{1}{2}]} \notin C([0,1])$. It isn't even enough to prove that there is no function in $C([0,1])$ which is equal almost everywhere to $\chi_{[0,\frac{1}{2}]}$. One must show that $(f_n)_{n\in\mathbb{N}}$ does not converge in $C([0,1])$. (Note, though, that this approach does work in the alternate solution, because the alternate solution explicitly uses the embedding of $C([0,1])$ in $L^2([0,1])$.)

An alternate proof of the triangle inequality can be derived from the fact that the norm defined on $C([0,1])$ comes from a scalar product.

Alternate solution. There is an obvious linear map $I\colon C([0,1]) \to L^2([0,1])$.

We claim that I is injective. Suppose $I(f) = 0$. Then $f = 0$ almost everywhere with respect to m. For every nonempty open set $U \subset [0,1]$, we have $m(U) > 0$. So $W = \{x \in [0,1\colon f(x) \neq 0\}$ contains no nonempty open set. Since f is continuous, $W = \varnothing$. Thus f is the zero element of $C([0,1])$.

Since I is injective and linear, the formula $\|f\| = \|I(f)\|_{L^2([0,1])}$ is a norm on $C([0,1])$.

We know that the range of I is dense in $L^2([0,1])$. We claim that it is not all of $L^2([0,1])$. Given this, $C([0,1])$ is a nonclosed subspace of $L^2([0,1])$, and therefore not complete; this will finish the solution. Set $f = \chi_{[0,\frac{1}{2}]}$. If f were in the range of I, then there would be $g \in C([0,1])$ such that $f = g$ almost everywhere. But this is clearly impossible. □

5. Let X, Y, and Z be Banach spaces. Let $S\colon Y \to Z$ be a bounded injective linear operator and let $T\colon X \to Y$ be a linear operator. Suppose $S \circ T\colon X \to Z$ is bounded. Prove that T is bounded. (Hint: use the Closed Graph Theorem.)

Solution. We prove that T has closed graph and then apply the Closed Graph Theorem to conclude that T is bounded. To show that T has closed graph, it is sufficient to prove that whenever $(\xi_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty} \xi_n = 0$ and $\lim_{n\to\infty} T\xi_n = \eta$, then $\eta = 0$.

Since S is bounded, we have $\lim_{n\to\infty} S(T\xi_n) = S\eta$. Since $S \circ T$ is bounded, we have $\lim_{n\to\infty} (S \circ T)\xi_n = 0$. Hence $S\eta = 0$, and injectivity of S implies that $\eta = 0$. □

6. Let m be Lebesgue measure on $[0, 2018]$. For $f \in L^2([0, 2018])$, let $T_f\colon L^2([0, 2018]) \to \mathbb{C}$ be the linear functional

$$T_f(g) = \int_{[0,\,2018]} fg \, dm$$

for $g \in L^2([0, 2018])$. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions in $L^2([0, 2018])$ such that $\|f_n\|_{L^2([0,\,2018])} \leq 1$ for each $n \in \mathbb{N}$. Prove that if $(f_n)_{n\in\mathbb{N}}$ converges to zero almost everywhere, then for each $g \in L^2([0, 2018])$,

$$\lim_{n\to\infty} T_{f_n}(g) = 0.$$

(In other words, prove that $(f_n)_{n\in\mathbb{N}}$ converges weakly to zero in $L^2([0, 2018])$).

Solution. To simplify the notation, set $I = [0, 2018]$ and abbreviate $\| \cdot \|_{L^2([0, 2018])}$ to $\| \cdot \|$.

Let $\epsilon > 0$. Since $|g|^2 \in L^1(I)$, there exists $\delta > 0$ such that for any measurable set $A \subset I$ with $m(A) < \delta$,

$$\int_A |g|^2 \, dm < \frac{\epsilon^2}{4}.$$

Set

$$\epsilon_0 = \frac{\epsilon}{2\sqrt{2018}(\|g\| + 1)}.$$

For $n \in \mathbb{N}$ define

$$E_n = \{ x \in I \colon |f_n(x)| > \epsilon_0 \} \qquad \text{and} \qquad T_n = \bigcup_{k=n}^{\infty} E_k.$$

Define $Z = \bigcap_{n=1}^{\infty} T_n$. Then $f_n \to 0$ almost everywhere implies $m(Z) = 0$. Since $m(I) < \infty$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $m(T_n) < \delta$.

Suppose that $n \geq N$. Then, using $\|f_n\| \leq 1$ and $m(T_n) < \delta$ at the second step,

$$\int_{T_n} |f_n g| \, dm \leq \left(\int_{T_n} |f_n|^2 \, dm \right)^{1/2} \left(\int_{T_n} |g|^2 \, dm \right)^{1/2} < \frac{\epsilon}{2}.$$

Also, using $|f_n| \leq \epsilon_0$ on $I \setminus T_n$ at the second step,

$$\int_{I \setminus T_n} |f_n g| \, dm \leq \left(\int_{I \setminus T_n} |f_n|^2 \, dm \right)^{1/2} \left(\int_{I \setminus T_n} |g|^2 \, dm \right)^{1/2}$$
$$\leq \epsilon_0 m(I \setminus T_n)^{1/2} \left(\int_I |g|^2 \, dm \right)^{1/2} \leq \epsilon_0 m(I)^{1/2} \|g\| < \frac{\epsilon}{2}.$$

Therefore

$$|T_{f_n}(g)| \leq \int_I |f_n g| \, dm < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the solution. ∎

We can also use Egoroff's Theorem. The argument involving the sets E_n in the first solution is part of the proof of Egoroff's Theorem, but one sees from that solution that we don't need the full strength of Egoroff's Theorem.

Alternate solution. To simplify the notation, set $I = [0, 2018]$ and abbreviate $\| \cdot \|_{L^2([0, 2018])}$ to $\| \cdot \|$.

Let $\epsilon > 0$. Choose $\delta > 0$ as in the first solution. Use Egoroff's Theorem to find a subset $B \subset I$ such that $m(I \setminus B) < \delta$ and $f_n \to 0$ uniformly on B. Choose $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sup_{x \in B} |f_n(x)| < \frac{\epsilon}{2\sqrt{2018}(\|g\| + 1)}.$$

For $n \geq N$, we have

$$|T_{f_n}(g)| \leq \int_I |f_n g| \, dm = \int_{I \setminus B} |f_n g| \, dm + \int_B |f_n g| \, dm.$$

The first integral on the right is less than $\frac{\epsilon}{2}$ by the argument used in the first solution for $\int_{T_n} |f_n g| \, dm$, and the second integral on the right is less than $\frac{\epsilon}{2}$ by the argument used in the first solution for $\int_{I \setminus T_n} |f_n g| \, dm$. So $|T_{f_n}(g)| < \epsilon$. ∎

7. Find all entire functions f such that $|f(z)| \leq |z|^{5/2}$ for all $z \in \mathbb{C}$.

Solution. We show that the only such function is the constant function $f(z) = 0$ for all $z \in \mathbb{C}$.

First, clearly $f(0) = 0$. Suppose f is not identically zero. Then there are an entire function h with $h(0) \neq 0$ and $k \in \mathbb{N}$ such that $f(z) = z^k h(z)$ for all $z \in \mathbb{C}$.

We claim that $k \geq 3$. To prove the claim, choose $\delta > 0$ such that $|h(z) - h(0)| < |h(0)|/2$ for all $z \in \mathbb{C}$ with $|z| < \delta$. Then for $|z| < \delta$, we have $|h(z)| > |h(0)|/2$, so

$$|z^k| \leq \frac{2|z^k h(z)|}{|h(0)|} \leq \frac{2|z|^{5/2}}{|h(0)|}.$$

For this to be true for all $z \in \mathbb{C}$ with $|z| < \delta$, we must have $k \geq 5/2$. Since $k \in \mathbb{N}$, it follows that $k \geq 3$.

Given the claim, whenever $|z| \geq 1$,

$$|h(z)| = \frac{|f(z)|}{|z^k|} \leq \frac{|f(z)|}{|z|^{5/2}} \leq 1.$$

Hence h is a bounded entire function, and h must be constant by Liouville's theorem. So $f(z) = h(0)z^k$ for all $z \in \mathbb{C}$. Then $|h(0)| \leq |z|^{5/2-k}$ for all $z \in \mathbb{C} \setminus \{0\}$. Since $k \geq 3$, this contradicts $h(0) \neq 0$. Hence f is identically zero. □

Alternate solution. We show that the only such function is the constant function $f(z) = 0$ for all $z \in \mathbb{C}$. Set $g(z) = z^{-2}f(z)$ for $z \in \mathbb{C} \setminus \{0\}$. Then $|g(z)| \leq |z|^{1/2}$ for $z \in \mathbb{C} \setminus \{0\}$, so $\lim_{z\to 0} g(z) = 0$, Therefore g has a removable singularity at 0, that is, there is an entire function h such that $h(z) = g(z)$ for all $z \in \mathbb{C} \setminus \{0\}$. Moreover, $h(0) = \lim_{z\to 0} g(z) = 0$. Therefore there is an entire function k such that $h(z) = zk(z)$ for all $z \in \mathbb{C}$. That is, $f(z) = z^3k(z)$ for all $z \in \mathbb{C}$. Now $|z^3k(z)| \leq |z|^{5/2}$ for all $z \in \mathbb{C}$, so $|k(z)| \leq |z|^{-1/2}$ for all $z \in \mathbb{C} \setminus \{0\}$. Therefore $\lim_{z\to\infty} k(z) = 0$. Since k is entire, Liouville's theorem implies that $k(z) = 0$ for all $z \in \mathbb{C}$. Therefore $f(z) = 0$ for all $z \in \mathbb{C}$. □

8. Let $n \geq 1$, and let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ be a monic polynomial with complex coefficients. Prove that $\max_{|z|=1} |f(z)| \geq 1$.

Solution. Set $g(z) = 1 + a_{n-1}z + a_{n-2}z^2 + \cdots + a_0z^n$ for $z \in \mathbb{C}$. Then $g(0) = 1$ and $g(z) = z^n f(1/z)$ for $z \neq 0$. Suppose $\max_{|z|=1} |f(z)| < 1$. Then $\max_{|z|=1} |g(z)| < 1$. Hence g has a local maximum inside the open unit disk D. This forces g to be a constant. Hence $g = 1$. This contradicts $\max_{|z|=1} |g(z)| < 1$. □

Alternate solution. Suppose $\max_{|z|=1} |f(z)| < 1$. Then for $|z| = 1$ we have $|z^n - (z^n - f(z))| < |z^n|$. So Rouché's Theorem says that $z \mapsto z^n$ and $z \mapsto z^n - f(z)$ have the same number of zeros in $D = \{z \in \mathbb{C} \colon |z| < 1\}$, counting multiplicity. But $z \mapsto z^n$ has n zeros in D, and $z \mapsto z^n - f(z)$, being a polynomial of degree at most $n - 1$, has at most $n - 1$ zeros in the whole complex plane. This contradiction shows that $\max_{|z|=1} |f(z)| \geq 1$. □

9. Suppose $f \in L^1((0, \infty))$ (using Lebesgue measure). Prove that

$$F(z) = \int_0^\infty f(t)e^{itz}\, dt$$

defines a holomorphic function on $\{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}$.

Solution. Set $\Omega = \{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}$. We prove that if $a \in \Omega$ then

$$\lim_{z\to a} \frac{F(z) - F(a)}{z - a}$$

exists, by showing that it is equal to

$$\int_0^\infty f(t)ite^{ita}\, dt.$$

It suffices to let $(z_n)_{n\in\mathbb{N}}$ be any sequence in $\Omega \setminus \{a\}$ such that $\lim_{n\to\infty} z_n = a$, and prove that

$$\lim_{z_n\to a} \frac{F(z_n) - F(a)}{z_n - a} = \int_0^\infty f(t)ite^{ita}\, dt. \tag{1}$$

We will use the Lebesgue Dominated Convergence Theorem.

For $n \in \mathbb{N}$ define $g_n \colon (0, \infty) \to \mathbb{C}$ by

$$g_n(t) = \frac{e^{itz_n} - e^{ita}}{z_n - a}$$

for $t \in (0, \infty)$. Then

$$\frac{F(z_n) - F(a)}{z_n - a} = \int_0^\infty f(t)g_n(t)\, dt$$

for all $n \in \mathbb{N}$. Since for each $t \in (0, \infty)$ the function $z \mapsto e^{itz}$ is a holomorphic function with derivative $z \mapsto ite^{itz}$, we have $\lim_{n\to\infty} g_n(t) = ite^{ita}$ for all $t \in (0, \infty)$. So $\lim_{n\to\infty} f(t)g_n(t) = f(t)ite^{ita}$ for all $t \in (0, \infty)$.

For $z \in \mathbb{C} \setminus \{0\}$ and $t \in (0, \infty)$, we estimate:

$$\left| \frac{e^{itz} - 1}{z} \right| = \left| \sum_{k=1}^{\infty} \frac{(it)^k z^{k-1}}{k!} \right| \leq t \sum_{k=1}^{\infty} \frac{t^{k-1} |z|^{k-1}}{k!} \leq t \sum_{k=1}^{\infty} \frac{t^{k-1} |z|^{k-1}}{(k-1)!} = t e^{t|z|}. \tag{2}$$

Set $b = \mathrm{Im}(a)$. Then $b > 0$. Choose $N \in \mathbb{N}$ so large that $n \geq N$ implies $|z_n - a| < b/2$. For such n, (2) implies

$$\left| \frac{e^{it(z_n - a)} - 1}{z_n - a} \right| \leq t e^{bt/2}, \tag{3}$$

so

$$|g_n(t)| = |e^{ita}| \left| \frac{e^{it(z_n - a)} - 1}{z_n - a} \right| \leq t e^{-bt/2}. \tag{4}$$

The function $t \mapsto t e^{-bt/2}$ is bounded on $(0, \infty)$, so $t \mapsto |f(t)| t e^{-bt/2}$ is integrable on $(0, \infty)$. By (4), we have $|f(t) g_n(t)| \leq |f(t)| t e^{-bt/2}$ for all $n \geq N$ and all $t \in (0, \infty)$, and we already saw that $\lim_{n \to \infty} f(t) g_n(t) = f(t) i t e^{ita}$ for all $t \in (0, \infty)$, so (1) follows from the Lebesgue Dominated Convergence Theorem. This completes the solution. □

Alternate solution. This solution differs only in the method used to get the estimate (3).

As in the first solution, set $b = \mathrm{Im}(a) > 0$, and choose $N \in \mathbb{N}$ so large that $n \geq N$ implies $|z_n - a| < b/2$. For such n, we have $\mathrm{Im}(z_n) > b/2$, so

$$\mathrm{Im}(z_n - a) = \mathrm{Im}(z_n) - \mathrm{Im}(a) > -\frac{b}{2}.$$

Further let $t \in (0, \infty)$. Then, if $\alpha \in [0, 1]$, we have

$$\big|\exp(it\alpha(z_n - a))\big| = \exp\big(\mathrm{Re}(it\alpha(z_n - a))\big) = \exp\big(-t\alpha \mathrm{Im}(z_n - a)\big) < \exp\big(t\alpha b/2\big) \leq e^{bt/2}.$$

Setting $h_t(z) = e^{itz}$ for $z \in \mathbb{C}$, we then have

$$\begin{aligned} \big|e^{it(z_n - a)} - 1\big| &= |h_t(z) - h_t(0)| \leq |z_n - a| \sup_{\alpha \in [0,1]} \big|h_t'(\alpha(z_n - a))\big| \\ &= |z_n - a| \sup_{\alpha \in [0,1]} \big|it \exp(it\alpha(z_n - a))\big| \leq t |z_n - a| e^{bt/2}. \end{aligned}$$

The estimate (3) follows. □

One can also solve this problem by combining Fubini's Theorem and Morera's Theorem. This is in principle a neat way to do it. Unfortunately, one must prove that F is continuous in order to use Morera's Theorem.

Second alternate solution. Set $\Omega = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$.

We first claim that F is continuous on Ω. To prove the claim, let $z \in \Omega$, and let $(z_n)_{n \in \mathbb{N}}$ be any sequence in Ω such that $\lim_{n \to \infty} z_n = z$. For $n \in \mathbb{N}$ and $t \in (0, \infty)$, set $g_n(t) = f(t) e^{itz_n}$. Then $\lim_{n \to \infty} g_n(t) = f(t) e^{itz}$ for all $t \in (0, \infty)$. For all $n \in \mathbb{N}$ and $t \in (0, \infty)$, using $\mathrm{Im}(z_n) > 0$ at the last step, we have

$$|e^{itz_n}| = \exp(\mathrm{Re}(itz_n)) = \exp(-t \mathrm{Im}(z_n)) \leq 1.$$

Therefore $|g_n(t)| \leq |f(t)|$. Since f is integrable on $(0, \infty)$, the Lebesgue Dominated Convergence Theorem implies that

$$F(z_n) = \int_0^{\infty} f(t) e^{itz_n} \, dt \to \int_0^{\infty} f(t) e^{itz} \, dt = F(z)$$

as $n \to \infty$. The claim is proved.

Now let γ be any triangle in Ω. We show that $\int_{\gamma} F(z) \, dz = 0$, by using Fubini's Theorem and the fact that the integrand in the definition of F is holomorphic as a function of z. Assume that γ is defined on $[a, b]$. Then, by definition,

$$\int_{\gamma} F(z) \, dz = \int_a^b F(\gamma(s)) \gamma'(s) \, ds.$$

Define a function $g \colon [a, b] \times (0, \infty) \to \mathbb{C}$ by

$$g(s, t) = f(t) \exp(it\gamma(s)) \gamma'(s).$$

The function $(s, t) \mapsto \exp(it\gamma(s))\gamma'(s)$ is continuous except on the product of a finite subset of $[a, b]$ with $(0, \infty)$. Therefore it is measurable. The function $(s, t) \mapsto f(t)$ is clearly measurable. Therefore g is the product of two measurable functions, hence measurable. So $|g|$ is measurable. (Checking that $|g|$ is measurable is an essential part of the solution.) We can now use Fubini's Theorem for nonnegative functions to calculate

$$\int_{[a,b]\times(0,\infty)} |g| \, d(m \times m) = \int_a^b \left(\int_0^\infty \big| f(t) \exp(it\gamma(s))\gamma'(s) \big| \, dt \right) ds = \int_a^b \|f\|_1 |\gamma'(s)| \, ds = \|f\|_1 \int_a^b |\gamma'(s)| \, ds,$$

which is finite. Therefore Fubini's Theorem for integrable functions can be applied, to get

$$\begin{aligned} \int_\gamma F(z) \, dz &= \int_a^b \left(\int_0^\infty g(s,t) \, dt \right) ds \\ &= \int_0^\infty f(t) \left(\int_a^b \exp(it\gamma(s))\gamma'(s) \, ds \right) dt = \int_0^\infty f(t) \left(\int_\gamma \exp(itz) \, dz \right) dt. \end{aligned}$$

Since Ω is convex, Cauchy's Theorem implies that the inner integral on the right is zero for all $t \in (0, \infty)$. Therefore $\int_\gamma F(z) \, dz = 0$.

Since γ is an arbitrary triangle in Ω, Morera's Theorem now implies that F is holomorphic. □