ANALYSIS QUALIFYING EXAM, FALL 2018

Instructions: *This is a 3 hour exam. Each problem is worth 10 points. Your proofs and solutions should be legible, complete, and correct to receive full credit. Do not write anything less than 1/4 inch from any edge of your paper.*

Notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$. (That is, $0 \notin \mathbb{N}$.)

1. Let $(X, \mathfrak{M})$ be a measurable space with a σ-finite positive measure μ. Prove that there is a finite positive measure ν on $(X, \mathfrak{M})$ such that $\mu \ll \nu$ and $\nu \ll \mu$. (Recall that $\mu \ll \nu$ means that μ is absolutely continuous with respect to ν, that is, for any $E \in \mathfrak{M}$, $\nu(E) = 0$ implies $\mu(E) = 0$).

2. Let (X, μ) be a measure space with $\mu(X) < \infty$. For $f \in L^2(X, \mu)$, prove that $|f| \log(|f|)$ is in $L^1(X, \mu)$. (Take $y \log(y) = 0$ when $y = 0$.) If $(X, \mu) = (\mathbb{R}, m)$, is it still the case that $f \in L^2$ implies $|f| \log(|f|) \in L^1(X, \mu)$?

3. Let $f \colon [0, 1] \to \mathbb{R}$ be a continuous function.

(a) Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx = 0$. (3 points).

(b) Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 (n + 1)x^n f(x)\, dx = f(1)$. (7 points)

4. Let $C([0, 1])$ be the space of continuous functions on $[0, 1]$. Prove that

$$\|f\| = \left(\int_{[0,1]} |f|^2\, dm \right)^{1/2}$$

defines a norm on $C([0, 1])$. Is $C([0, 1])$ a Banach space with respect to this norm? Justify your answer.

5. Let X, Y, and Z be Banach spaces. Let $S \colon Y \to Z$ be a bounded injective linear operator and let $T \colon X \to Y$ be a linear operator. Suppose $S \circ T \colon X \to Z$ is bounded. Prove that T is bounded. (Hint: use the Closed Graph Theorem.)

6. Let m be Lebesgue measure on $[0, 2018]$. For $f \in L^2([0, 2018])$, let $T_f \colon L^2([0, 2018]) \to \mathbb{C}$ be the linear functional

$$T_f(g) = \int_{[0,\,2018]} fg\, dm$$

for $g \in L^2([0, 2018])$. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions in $L^2([0, 2018])$ such that $\|f_n\|_{L^2([0,\,2018])} \le 1$ for each $n \in \mathbb{N}$. Prove that if $(f_n)_{n\in\mathbb{N}}$ converges to zero almost everywhere, then for each $g \in L^2([0, 2018])$,

$$\lim_{n\to\infty} T_{f_n}(g) = 0.$$

(In other words, prove that $(f_n)_{n\in\mathbb{N}}$ converges weakly to zero in $L^2([0, 2018])$).

7. Find all entire functions f such that $|f(z)| \le |z|^{5/2}$ for all $z \in \mathbb{C}$.

8. Let $n \ge 1$, and let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial with complex coefficients. Prove that $\max_{|z|=1} |f(z)| \ge 1$.

9. Suppose $f \in L^1((0, \infty))$ (using Lebesgue measure). Prove that

$$F(z) = \int_0^\infty f(t)e^{itz}\, dt$$

defines a holomorphic function on $\{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}$.