ANALYSIS QUALIFYING EXAM, WINTER 2018

Instructions: *This is a 3 hour exam. Each problem is worth 10 points. Your proofs and solutions should be legible, complete, and correct to receive full credit. Do not write anything less than 1/4 inch from any edge of your paper.*

Notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$. (That is, $0 \notin \mathbb{N}$.)

1. Suppose f is a Lebesgue measurable function on $\mathbb{R}$ such that f and $x \mapsto xf(x)$ are both in $L^2(\mathbb{R})$. Prove that $f \in L^1(\mathbb{R})$.

2. Let f be a nonnegative Lebesgue measurable function on $[0, 1]$ such that $f > 0$ almost everywhere. Prove that for any $\epsilon > 0$ there is $\delta > 0$ such that for any Lebesgue measurable subset E of $[0, 1]$ with $m(E) > \epsilon$, we have $\int_E f\, dm > \delta$.

3. Let m be Lebesgue measure on $[0, 1]$. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of strictly positive measurable functions on $[0, 1]$ such that

$$\lim_{n\to\infty} \int_{[0,1]} f_n^2\, dm = 0.$$

Prove that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n \log(f_n)|\, dm = 0.$$

4. For which $p \in [1, \infty]$ is there a nonzero bounded linear functional T on $L^p([0, 1])$ such that T vanishes on the subspace $C([0, 1]) \subset L^p([0, 1])$? Justify your answer.

5. Let l^1 be the Banach space of complex sequences defined as follows:

$$a = (a_1, a_2, a_3, \ldots) \in l^1 \text{ if and only if } \|a\|_{l^1} = \sum_{k=1}^{\infty} |a_k| \text{ is finite.}$$

Let X be a separable Banach space. Suppose $\{x_k : k \in \mathbb{N}\}$ is a countable dense subset of the closed unit ball in X. Define a linear operator $S\colon l^1 \to X$ by

$$Sa = \sum_{k=1}^{\infty} a_k x_k$$

for $a \in l^1$. Prove that S is bounded and surjective.

6. For $t \in \mathbb{R}$, let $T_t\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator $(T_t f)(x) = f(x - t)$ for $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. Prove that $\|T_t - T_s\| \geq 2$ for $t \neq s$, where the norm is the operator norm.

7. Suppose $f\colon D \to \mathbb{C}$ is a holomorphic function on the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. If f is injective on $D \setminus \{0\}$, prove that f is injective on D.

8. Suppose f is an entire function such that $|f(z)| \leq 1 + \sqrt{|z|}$ for all $z \in \mathbb{C}$. Prove that f is constant.

9. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos(\theta)}$.