Fall 2019 Algebra Qualifying Exam

THREE HOURS. Attempt *at most five* out of the following six questions.

1. Let V be a 5-dimensional vector space over a field F.

   (a) Let $T : V \to V$ be a linear transformation with characteristic polynomial $(x-1)^3(x-2)^2$ and minimal polynomial $(x-1)^2(x-2)$.

      i. Write down a matrix which represents T in Jordan normal form.

      ii. Write down the matrix which represents T in rational normal form.

   (b) Instead, let $T : V \to V$ be a *nilpotent* linear transformation which has exactly one 2-dimensional invariant subspace.

      i. How many similarity classes of such linear maps T are there?

      ii. Assuming finally that F is the finite field $\mathbb{F}_q$ with q elements, find an explicit formula for the number of such linear maps T.

2. Let A be a finite-dimensional algebra over a field F equipped with a non-degenerate symmetric bilinear form $(\cdot,\cdot) : A \times A \to F$. Let $x_1, \ldots, x_n$ be a basis for A and $y_1, \ldots, y_n$ be the dual basis with respect to the given form, i.e., $(x_i, y_j) = \delta_{i,j}$ for all $i, j = 1, \ldots, n$.

   (a) Show that the element
   $$z := \sum_{i=1}^{n} x_i y_i$$
   is well defined independent of the initial choice of the basis $x_1, \ldots, x_n$.

   (b) Assume for the remainder of the question that the form $(\cdot,\cdot)$ is *invariant*, which means that $(ab, c) = (a, bc)$ for all $a, b, c \in A$. Show that $([a,b], c) = (a, [b,c])$ where $[\cdot,\cdot]$ is the commutator.

   (c) Let $a \in A$ be any element and suppose that $[a, x_i] = \sum_{j=1}^{n} \lambda_{ij} x_j$ and $[a, y_i] = \sum_{j=1}^{n} \mu_{ij} y_j$ for scalars $\lambda_{ij}, \mu_{ij} \in F$. Show that $\lambda_{ij} + \mu_{ji} = 0$. Deduce that z lies in the *center* of the algebra A. (You may find the identity $[a, xy] = [a,x]y + x[a,y]$ helpful here.)

3. In this question, R is a ring and $e \in R$ is an idempotent, so that eRe is another ring with identity element e.

   (a) What does it mean to say that R is *semisimple*? State the Artin-Wedderburn Theorem.

   (b) If V is a completely reducible R-module of finite length, show that the algebra $\mathrm{End}_R(V)$ is semisimple. Deduce for a semisimple ring R that eRe is semisimple too.

   (c) Assuming that R is left Artinian, show that $J(eRe) = eJ(R)e$, where J denotes Jacobson radical.

4. Let G be a finite group. Adopt the usual notation for the character table of G. In particular, $C_1 = \{1\}, C_2, \ldots, C_n$ are the conjugacy classes and $\chi_1 = \mathbf{1}, \chi_2, \ldots, \chi_n$ are the irreducible characters.

   (a) Let $\rho : G \to GL_n(\mathbb{C})$ be a finite-dimensional representation with associated character χ. Prove that $\ker \rho = \{g \in G \mid \chi(g) = \chi(1)\}$.

   (b) Use the row and column orthogonality relations to work out the values of α, β, γ and δ in the following character table:

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	-2	-1	γ	$-\beta$	$-\alpha$	1	α	β
χ_3	2	-2	-1	γ	$-\alpha$	$-\beta$	1	β	α
χ_4	3	3	0	-1	β	α	0	α	β
χ_5	3	3	0	-1	α	β	0	β	α
χ_6	4	-4	1	γ	-1	-1	-1	1	1
χ_7	4	4	1	γ	-1	-1	1	-1	-1
χ_8	5	5	-1	1	0	0	-1	0	0
χ_9	δ	$-\delta$	0	γ	1	1	0	-1	-1

   (c) Let G be a group with the character table computed in (b). Work out the character table of the group $H = G/Z(G)$, explaining your steps. What group is H?

5. The dihedral group $G = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$ acts on the $\mathbb{C}$-algebra $S = \mathbb{C}[x, y]$ by algebra automorphisms so that

$$a \cdot x = \omega x, \qquad a \cdot y = \omega^{-1} y, \qquad b \cdot x = y,$$

   where $\omega = e^{2\pi i/3}$. Let $R := S^G$ be the invariant subalgebra.

   (a) Show that $R = \mathbb{C}[x^3 + y^3, xy]$.

   (b) Show that $R \subseteq S$ is an integral extension.

   (c) Find an explicit monic polynomial $f(t) \in R[t]$ such that $f(x) = 0$.

6. Work over an algebraically closed field F of characteristic zero.

   (a) Let X be an affine variety with coordinate algebra $F[X]$. State the *Nullstellensatz*. Then use it to show that a subset $S \subseteq X$ is dense (in the Zariski topology) if and only if the following property holds for all $f \in F[X]$:

$$(f(s) = 0 \text{ for all } s \in S) \Rightarrow f = 0.$$

   (b) Given affine varieties X and Y and dense subsets $S \subseteq X$ and $T \subseteq Y$, prove that $S \times T$ is dense in $X \times Y$.

   (c) Show that the integer lattice $\mathbb{Z}^n$ is dense in F^n.