Topology Qualifying Exam
Fall 2018

Name: ______________________________

1. Find three composites of normal covering maps $X_2 \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0$ (with spaces which may differ) such that the group of deck transformations for p_1 and p_2 are each $\mathbb{Z}/4$ and the group of deck transformations for $p_1 \circ p_2$ is $\mathbb{Z}/4 \times \mathbb{Z}/4$, or $\mathbb{Z}/8 \times \mathbb{Z}/2$, or $\mathbb{Z}/16$, respectively.

2. Recall that the simplex Δ^n is the convex span of the standard basis for $\mathbb{R}^{n+1}$, which we choose to call $e_0, \ldots, e_n$. We use the notation $[v_0, \ldots, v_k]$ for the convex span of the basis vectors $e_{v_0}, \ldots, e_{v_k}$, which is a k-simplex in $\Delta^n = [0, \ldots, n]$.

   Compute the homology groups of the space X obtained as a quotient of Δ^3 by setting $[v_0, \ldots, v_k] \sim [w_0, \ldots, w_k]$ when every v_i is congruent to w_i modulo two.
   Clarification: here we allow for the vertices to not occur in order. Suggestion: it could be helpful to use labels by mod-two residues.

3. Show that homotopic maps induce the same map on singular homology. (You may assume basic results about chain complexes.)

4. Is there a connected product $X = Y \times Z$, where both Y and Z have non-trivial reduced homology, with the following homology? Explain.

   - $H_1(X) \cong \mathbb{Z}/4$
   - $H_2(X) \cong \mathbb{Z}/5$
   - $H_3(X) \cong \mathbb{Z}/6$
   - $H_4(X) \cong \mathbb{Z}/2$ and
   - $H_5(X) \cong \mathbb{Z}/2$.

5. Let X be a finite Δ-complex, based at some point x_0 . Briefly explain whether each of the following is a based homotopy invariant, only a based homeomorphism invariant but not homotopy invariant, or not a homeomorphism invariant of the underlying space. (The length of explanation needed for each statement can vary.)

   1. $\sum \text{rank}_{\mathbb{Q}}\, C_i^{\Delta}(X;\mathbb{Q})$
   2. $\sum (-1)^i \text{rank}_{\mathbb{Q}}\, C_i^{\Delta}(X;\mathbb{Q})$
   3. $H_n(X, x_0)$, for all n.
   4. $H_n(X, X - x_0)$, where $X - x_0$ denotes the complement of x_0 in X, for all n.

   For the last two parts, an example for a single n would suffice to establish non-invariance.

6. Let $X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to \cdots$ be a directed system of spaces where the map ι_n from each X_n to the direct limit is a good inclusion (that is, the maps in the directed system are all inclusions, and each X_i has an open neighborhood in $\varinjlim X_j$ which deformation retracts onto it). Define a map from $\varinjlim H_d(X_i) \to H_d(\varinjlim X_i)$ and show that it is *injective*.

7. (a) Define a map from $\mathbb{R}P^7$ to itself with no fixed point.

   (b) Show that any map from $X = \mathbb{R}P^7/\mathbb{R}P^1$ to itself must have a fixed point.

8. Find, with justification, two embeddings of $S^1 \bigsqcup S^1$ in $\mathbb{R}^3$ whose complements are not homotopy equivalent.

9. (a) Sketch briefly (2-4 sentences) an argument that homology groups vanish above the dimension of a CW complex, and an argument that homotopy groups do not.

   (b) State two other important differences between homotopy groups π_i and homology groups H_i for $i \geq 0$ and again give brief (1-4 sentence) justification.