Qualifying Exam in Algebra, Fall 2018

Part I. True or false. Justify your answer by giving a proof or counterexample. 10 points each.

1. For an algebraically closed field F, the abelian group $F^\times$ is injective.

Answer: TRUE. By Baer's criterion an abelian group is injective if and only if it is divisible. The group $F^\times$ is divisible since the equation $x^n = a$ has a solution for any $a \in F^\times$ and $n \in \mathbb{Z}_{>0}$ since the field F is algebraically closed.

2. There exists a nonzero natural transformation $\bigwedge^2 \to \mathrm{Id}$ where $\mathrm{Id} : V \mapsto V$ and $\bigwedge^2 : V \mapsto \bigwedge^2 V$ are functors $\underline{\mathrm{Vec}_\mathbb{C}} \to \underline{\mathrm{Vec}_\mathbb{C}}$.

Answer: FALSE. Assume for a contradiction that there is a non-zero natural transformation η. Pick V and $x \in \bigwedge^2 V$ so $v := \eta_V(x) \neq 0$. Let $f : V \to \mathbb{C}$ be a linear map that is non-zero on v. Looking at the naturality square associated to f gives a contradiction because one way around the square is non-zero and the other way is zero as $\bigwedge^2(\mathbb{C}) = 0$.

3. A field extension of degree 4 has an intermediate subfield of degree 2.

Answer: FALSE. Let E/F be a Galois extension with $\mathrm{Gal}(E/F) = S_4$, e.g. $E = \mathbb{C}(x_1, x_2, x_3, x_4)$ and $F = \mathbb{C}(x_1, x_2, x_3, x_4)^{S_4}$. Let K be the intermediate field corresponding to $S_3 \subset S_4$, i.e. $K = E^{S_3}$. Then $|K : F| = [S_4 : S_3] = 4$ and there is no subfield $F \subset L \subset K$ with $|L : F| = 2$ since it would correspond to a subgroup $H \subset S_4$ of index 2 and containing S_3. Such subgroup does not exist: the only subgroup of S_4 of index 2 is A_4 and it does not contain S_3.

4. The ideal $I = (x^3 - xyz, yz) \triangleleft \mathbb{C}[x, y, z]$ is an intersection of some prime ideals.

Answer: FALSE. If P is prime containing I, then $x^3 \in I$ so $x^3 \in P$ so $x \in P$. So if I was an intersection of prime ideals, it would contain x, which it does not.

5. Let $S \to R$ be a (unital) ring homomorphism. For a finitely generated projective $S-$module P, the $R-$module $R \otimes_S P$ is finitely generated projective.

Answer: TRUE. The module P is a direct summand of free module S^n for some $n \in \mathbb{Z}_{>0}$. Thus $R \otimes_S P$ is a direct summand of $R \otimes_S S^n = R^n$, so it is finitely generated and projective.

Part II. Longer problems. 10 points each.

1. Prove that the group $GL_6(\mathbb{F}_2)$ has an element of order 63.

Solution: The field $\mathbb{F}_{64}$ is a vector space of dimension 6 over $\mathbb{F}_2$. The multiplication by an element of $\mathbb{F}_{64}$ is a linear map in this vector space. Thus we have an embedding of groups $\mathbb{F}_{64}^\times \subset GL_6(\mathbb{F}_2)$. The group $\mathbb{F}_{64}^\times$ is cyclic of order 63, so the image of a generator is a desired element.

2. Let A be an abelian group and let $a \in A$. Prove that the order of a is infinite if and only if $a \otimes 1 \in A \otimes_{\mathbb{Z}} \mathbb{Q}$ is nonzero.

Solution: Assume that the order of a is finite, that is $na = 0$ for some $n \in \mathbb{Z}_{>0}$. Then $a \otimes 1 = a \otimes (n \cdot \frac{1}{n}) = na \otimes \frac{1}{n} = 0$.

Now assume that the order of a is infinite. Let $\mathbb{Z} \to A$ be an injective homomorphism sending $1 \in \mathbb{Z}$ to $a \in A$. Recall that $\mathbb{Q}$ is a flat $\mathbb{Z}-$module (say as a localization of commutative ring $\mathbb{Z}$). Thus tensoring by $\mathbb{Q}$ an exact sequence

$$0 \to \mathbb{Z} \to A \to A/\mathbb{Z} \to 0$$

we get an exact sequence

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \to A \otimes_{\mathbb{Z}} \mathbb{Q} \to A/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \to 0.$$

The first map sends $1 \otimes 1$ to $a \otimes 1$. Note that $1 \otimes 1 \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ is nonzero as it maps to $1 \in \mathbb{Q}$ under the natural isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$. Thus $a \otimes 1 \neq 0$ as an image of nonzero element under an injective homomorphism.

3. For a point $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ let $I_a = (x_1 - a_1, \ldots, x_n - a_n) \lhd \mathbb{C}[x_1, \ldots, x_n]$ be the corresponding maximal ideal. Prove that a subset $S \subset \mathbb{C}^n$ is dense in Zariski topology if and only if $\bigcap_{s \in S} I_s = 0$.

Solution: A subset of a topological space is dense if and only if it is not contained in a proper closed subset. Any closed subset of $\mathbb{C}^n$ is $\mathcal{V}(I)$ for some ideal $I \lhd \mathbb{C}[x_1, \ldots, x_n]$. Such subset $\mathcal{V}(I)$ is proper if and only if $I \neq 0$: if $0 \neq f \in I$ then there is a point $x \in \mathbb{C}^n$ such that $f(x) \neq 0$. We have $S \subset \mathcal{V}(I)$ if and only if $f(s) = 0$ for any $s \in S$, $f \in I$ or, equivalently, $I \subset \bigcap_{s \in S} I_s = 0$. The result follows.

4. Suppose that an irreducible character of a finite group takes precisely two values. Prove that these values are ± 1.

Solution: Without loss of generality we can replace the group by its image in the irreducible representation in question. Thus we can and will assume that the irreducible character χ in question is faithful. Hence $\chi(g) = \chi(1)$ if and only if $g = 1$. Thus a suitable linear combination of the trivial character and χ takes value 0 on all elements $g \neq 1$ of the group. Thus this linear combination is proportional to the regular representation. Thus our group has just two irreducible representations as all of them must appear in the regular representation with nonzero multiplicity. It follows that $\chi(1) = 1$ (e.g. since $\chi(1)$ must divide $|G| = 1 + \chi(1)^2$) and $|G| = 2$. Thus χ is a homomorphism $G \simeq \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$ and the remaining value of χ is -1.

5. (i) (5 points) Prove that the ring $U_n(\mathbb{C})$ of upper triangular $n \times n$ matrices over $\mathbb{C}$ is isomorphic to its opposite ring.

(ii) (5 points) Show that the ring of 2×2 upper triangular matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, b \in \mathbb{C}$ and $d \in \mathbb{R}$ is not isomorphic to its opposite ring.

Solution: (a) The transpose map $X \mapsto X^T$ is an anti-isomorphism $U_n(\mathbb{C}) \to \tilde{U}_n(\mathbb{C})$ where $\tilde{U}_n(\mathbb{C})$ is the ring of lower triangular matrices. Thus the opposite ring of $U_n(\mathbb{C})$ is isomorphic to $\tilde{U}_n(\mathbb{C})$. Now the ring $\tilde{U}_n(\mathbb{C})$ is isomorphic to $U_n(\mathbb{C})$ via the map $X \mapsto w_0 X w_0^{-1}$ where w_0 is the matrix with 1's on the anti-diagonal and zeros elsewhere.

(b) Just as above the opposite ring is the ring of 2×2 lower triangular matrices $\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ with $a, b \in \mathbb{C}$ and $d \in \mathbb{R}$. The identity element of both rings decomposes as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

into a sum of orthogonal primitive idempotents (both idempotents are primitive even in a bigger ring $\mathrm{Mat}_2(\mathbb{C})$). This gives decompositions of the free modules of rank 1 over both rings into indecomposable summands:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

and

$$\begin{pmatrix} a & 0 \\ b & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}.$$

We see that the dimensions over $\mathbb{R}$ (which is the center of both rings) of the summands are 2 and 3 in one case and 4 and 1 in another case. Thus by the Krull-Schmidt theorem the rings can't be isomorphic.