Qualifying Exam in Algebra, Fall 2018

Part I. True or false. Justify your answer by giving a proof or counterexample. 10 points each.

1. For an algebraically closed field F, the abelian group $F^\times$ is injective.
2. There exists a nonzero natural transformation $\bigwedge^2 \to \mathrm{Id}$ where $\mathrm{Id} : V \mapsto V$ and $\bigwedge^2 : V \mapsto \bigwedge^2 V$ are functors $\underline{\mathrm{Vec}_{\mathbb{C}}} \to \underline{\mathrm{Vec}_{\mathbb{C}}}$.
3. A field extension of degree 4 has an intermediate subfield of degree 2.
4. The ideal $I = (x^3 - xyz, yz) \triangleleft \mathbb{C}[x, y, z]$ is an intersection of some prime ideals.
5. Let $S \to R$ be a (unital) ring homomorphism. For a finitely generated projective $S-$module P, the $R-$module $R \otimes_S P$ is finitely generated projective.

Part II. Longer problems. 10 points each.

1. Prove that the group $GL_6(\mathbb{F}_2)$ has an element of order 63.
2. Let A be an abelian group and let $a \in A$. Prove that the order of a is infinite if and only if $a \otimes 1 \in A \otimes_{\mathbb{Z}} \mathbb{Q}$ is nonzero.
3. For a point $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ let $I_a = (x_1 - a_1, \ldots, x_n - a_n) \triangleleft \mathbb{C}[x_1, \ldots, x_n]$ be the corresponding maximal ideal. Prove that a subset $S \subset \mathbb{C}^n$ is dense in Zariski topology if and only if $\bigcap_{s \in S} I_s = 0$.
4. Suppose that an irreducible character of a finite group takes precisely two values. Prove that these values are ± 1.
5. (i) (5 points) Prove that the ring $U_n(\mathbb{C})$ of upper triangular $n \times n$ matrices over $\mathbb{C}$ is isomorphic to its opposite ring.

(ii) (5 points) Show that the ring of 2×2 upper triangular matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, b \in \mathbb{C}$ and $d \in \mathbb{R}$ is not isomorphic to its opposite ring.