1. Differential Geometry Quals Problems for Fall 2018

Problem 1.1. A derivation of $C^\infty(\mathbb{R}^m)$ based at P is a linear map L from $C^\infty(\mathbb{R}^m)$ to $\mathbb{R}$ satisfying the Leibnitz rule

$$L(fg) = f(P)L(g) + L(f)g(P)\,.$$

Let L be a derivation of $C^\infty(\mathbb{R}^m)$ based at $P = 0$, and let $\vec{x} = (x^1, \cdots, x^m)$ be the coordinate functions on $\mathbb{R}^m$. Show there exist real constants $a^1, \ldots, a^m$ so that

$$L(f) = a^1 \frac{\partial f}{\partial x^1}(0) + \cdots + a^m \frac{\partial f}{\partial x^m}(0)\,.$$

Problem 1.2. Let $ds^2 := \frac{dx^2+dy^2}{f(x^2+y^2)}$ where $f(t)$ is a smooth positive function. Prove or disprove the assertion: "radial curves from the origin are unparameterized geodesics in this geometry".

Problem 1.3. Let $\mathcal{S} := \{\vec{x} : (x^1)^2 + \cdots + (x^m)^2 - (x^{m+1})^2 = -1\}$ be the pseudosphere in Minkowski space $\mathbb{R}^{m+1}$, where the ambient inner product is

$$\langle x, y\rangle = x^1y^1 + \cdots + x^my^m - x^{m+1}y^{m+1}\,.$$

The restriction of $\langle x, y\rangle$ to $\mathcal{S}$ induces a Riemannian metric on $\mathcal{S}$. Determine the sectional curvature of $\mathcal{S}$.

Problem 1.4. Let $\mathfrak{g}$ be the Lie algebra of a compact connected Lie group G. Then $\mathfrak{g}^*$ is the vector space of left-invariant 1-forms and $\Lambda^p(\mathfrak{g}^*)$ is the vector space of left-invariant p-forms. Let $H^p(\mathfrak{g}) := \frac{\ker\{d:\Lambda^p(\mathfrak{g}^*)\to\Lambda^{p+1}(\mathfrak{g}^*)\}}{\text{image}\{d:\Lambda^{p-1}(\mathfrak{g}^*)\to\Lambda^p(\mathfrak{g}^*)\}}$. The inclusion of $\Lambda^p(\mathfrak{g}^*)$ into $C^\infty(\Lambda^p(G))$ induces a natural map $\iota : H^p(\mathfrak{g}) \to H^p_{\text{DeR}}(G)$. Show that ι is an isomorphism.

Problem 1.5. Let

$$M = S^2 \times S^2 := \{(x, y, z, u, v, w) : x^2 + y^2 + z^2 = u^2 + v^2 + w^2 = 1\} \subset \mathbb{R}^6\,,$$

equipped with the metric induced from $\mathbb{R}^6$. Let

$$\begin{aligned}
\sigma(t) &:= (\cos(t), \sin(t), 0, \cos(t), \sin(t), 0),\\
e_1(t) &:= \tfrac{1}{\sqrt{2}}(-\sin(t), \cos(t), 0, -\sin(t), \cos(t), 0),\\
e_2(t) &:= \tfrac{1}{\sqrt{2}}(-\sin(t), \cos(t), 0, \sin(t), -\cos(t), 0),\\
e_3(t) &:= (0, 0, 1, 0, 0, 0),\\
e_4(t) &:= (0, 0, 0, 0, 0, 1).
\end{aligned}$$

(1) Show that σ is a geodesic.
(2) Show $\{e_1, e_2, e_3, e_4\}$ is a parallel orthonormal frame along σ.
(3) Determine the curvature tensor R_{ijkl} relative to this frame. You may use without proof the fact that the sectional curvature of S^2 is $+1$.
(4) Determine all the Jacobi vector fields along σ.

Problem 1.6. Cite carefully the theorems you use. An important part of this problem is establishing knowledge of some basic facts.

(1) Prove or disprove the following assertion: "There exists a Riemannian metric of non-positive sectional curvature on $S^1 \times S^2$."
(2) Prove or disprove the following assertion: "There exists a Riemannian metric on $S^1 \times S^2$ so that the Ricci curvature $\rho(X, X) > 0$ for every $0 \neq X \in T(S^1 \times S^2)$.

Problem 1.7. Let $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ with the product metric $ds^2 = d\theta_1^2 + d\theta_2^2$. For what values of ε is the geodesic ball of radius ε geodesically convex?

Problem 1.8. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\} \subset \mathrm{GL}(2, \mathbb{R})$ be the $ax + b$ group.

(1) Determine the Killing form of G.
(2) Show that G is not unimodular.

Problem 1.9. For each $n \geq 2$, construct a compact complex manifold M of complex dimension n which does not admit a Kähler metric.

Problem 1.10. Let $C(\phi) = (x(\phi), y(\phi))$ be a curve in the (x, y) plane which is parametrized by arc length, i.e. $\|\dot{C}\|^2 = 1$. Assume $y(\phi) > 0$ for all ϕ. Revolve C about the x-axis to form a surface $S \subset \mathbb{R}^3$, which we equip with the induced metric. $T(\phi, \theta) = (x(\phi), \cos(\theta)y(\phi), \sin(\theta)y(\phi))$ parametrizes S for $0 \leq \theta \leq 2\pi$.

(1) Determine when the curves $\sigma_\theta(\phi) := T(\phi, \theta)$ are geodesics.
(2) Determine when the curves $\tau_\phi(\theta) := T(\phi, \theta)$ geodesics.