Galois Representations in Étale Fundamental Groups and the Profinite Grothendieck-Teichmüller Group

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Abstract

It is well known that the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ behaves similarly to a fundamental group to the extent that it describes the finite covering space theory of $\mathbf{Spec}(\mathbb{Q})$. We will extend this analogy to the étale fundamental group of a scheme and explain how this can be used to produce Galois representations in étale fundamental groups of schemes over fields arising from geometric monodromy. In particular we will study a faithful Galois representation in the étale fundamental group of the projective line minus three points and its relation to Grothendieck's dessins. Moreover we will exhibit two relations on this Galois representation making it factor through the coarse profinite Grothendieck-Teichmüller group $\widehat{\mathbf{GT}}_0$, expressing the compatibility of the Galois action on dessins with certain recoloring and duality operations on dessins. Finally we will describe the profinite Grothendieck-Teichmüller group $\widehat{\mathbf{GT}}$ and some conjectures relating it to the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Contents

Introduction

The absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, despite its central importance in algebraic geometry and number theory, is poorly understood. It is a large and unwieldy profinite group which describes the rich Galois theory of number fields in the sense that its conjugacy classes of open subgroups are in bijection with isomorphism classes of number fields, or equivalently isomorphism classes of connected schemes finite étale over $\mathbf{Spec}(\mathbb{Q})$. In fact the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ can be understood as a fundamental group for $\mathbf{Spec}(\mathbb{Q})$, and it is no mistake that almost every notion in Galois theory can be described in the language of covering spaces: for instance the Galois action is the monodromy action on covering spaces, and Galois extensions of $\mathbb{Q}$ are equivalently those covering spaces whose monodromy is transitive; the Galois groups $\mathbf{Gal}(\mathbb{Q}^{(n)}/\mathbb{Q})$ of maximal algebraic extensions unramified away from n can be understood as a fundamental group for $\mathbf{Spec}(\mathbb{Z}[\frac{1}{n}])$, and in this sense a fundamental group for $\mathbf{Spec}(\mathbb{Z})$ is understood to be trivial since by Minkowski every number field is somewhere ramified.

Unfortunately the tautological action of the absolute Galois group on number fields tells us very little about the structure of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as a profinite group. For instance it is not clear from this description how one might describe the absolute Galois group in terms of generators and relations, or as a subgroup of some better understood profinite group; the tautological action of the absolute Galois group on number fields is far too self-referential for this purpose.

The goal of this paper is to describe a setting in which the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ can be studied in terms of its geometric action on curves over number fields rather than its tautological action on number fields themselves. In doing so we will construct a profinite group $\widehat{\mathbf{GT}}_0$ called the coarse Grothendieck-Teichmüller group having $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as a profinite subgroup, and we will discuss its refinement to a profinite group $\widehat{\mathbf{GT}}$ called the Grothendieck-Teichmüller group which is conjectured to be isomorphic to $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This gives at least a partial description of the absolute Galois group as a subgroup of some better understood profinite group: although both $\widehat{\mathbf{GT}}_0$ and $\widehat{\mathbf{GT}}$ are just as unwieldy as $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, they are defined independently from the Galois theory of number fields and hence serve to characterize the absolute Galois group in a non-tautological way.

In order to accomplish this we must define a scheme-theoretic analog of the fundamental group of topological spaces which is flexible enough to accommodate the Galois theory of fields as well as the geometric covering theory of varieties over algebraically closed fields. Specifically, for X a connected scheme and $\overline{x}$ a geometric point we will define a profinite group $\widehat{\pi}_1(X, \overline{x})$ called the étale fundamental group with the property that its conjugacy classes of open subgroups are in bijection with isomorphism classes of schemes finite étale over X. The étale fundamental group correctly generalizes both the absolute Galois group and what would be the geometric fundamental group, in the sense that for k a field with fixed separable closure $k^{\mathbf{s}}$ we have $\widehat{\pi}_1(\mathbf{Spec}(k), \overline{k}) \simeq \mathbf{Gal}(k^{\mathbf{s}}/s)$, and for X a scheme of finite type over k we have a short exact sequence

$$0 \to \widehat{\pi}_1(X_{\overline{k}}, \overline{x}) \to \widehat{\pi}_1(X, \overline{x}) \to \mathbf{Gal}(k^{\mathbf{s}}/k) \to 0$$

expressing the fact that schemes finite étale over X are obtained by extension of scalars, by finite geometric covering spaces of X, or some combination thereof. In particular when k is a field of characteristic 0 we will see that the étale fundamental group $\widehat{\pi}_1(X_{\overline{k}}, \overline{x})$ coincides with the profinite completion $\pi_1(X^{\mathbf{an}}, x)^\wedge$ of the topological fundamental group of the associated complex analytic space, and when k is a field of positive characteristic we will see that the étale fundamental group $\widehat{\pi}_1(X_{\overline{k}}, \overline{x})$ coincides with the étale fundamental group of a lift to characteristic 0 when completed away from the characteristic.

This étale homotopy exact sequence will be the main source for Galois representations in étale fundamental groups. Specifically for X such a scheme over a field k, the conjugation action of $\widehat{\pi}_1(X, \overline{x})$ on $\widehat{\pi}_1(X_{\overline{k}}, \overline{x})$ restricts to an inner action of $\widehat{\pi}_1(X_{\overline{k}}, \overline{x})$ on itself and quotients to an outer action of $\mathbf{Gal}(\overline{k}/k)$ on $\widehat{\pi}_1(X_{\overline{k}}, \overline{x})$, which we can express as a morphism of short exact sequences of profinite groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\pi}_1(X_{\overline{k}}, \overline{x}) & \longrightarrow & \widehat{\pi}_1(X, \overline{x}) & \underset{\overset{s_x}{\longleftarrow}}{\longrightarrow} & \mathbf{Gal}(\overline{k}/k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \overset{\rho_x}{\swarrow} & \downarrow \rho_{\text{out}} & & \\
0 & \longrightarrow & \mathbf{Inn}(\widehat{\pi}_1(X_{\overline{k}}, \overline{x})) & \longrightarrow & \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{k}}, \overline{x})) & \longrightarrow & \mathbf{Out}(\widehat{\pi}_1(X_{\overline{k}}, \overline{x})) & \longrightarrow & 0
\end{array}$$

In particular given a choice of section $s_x : \mathbf{Gal}(\overline{k}/k) \to \widehat{\pi}_1(X, \overline{x})$ of the étale homotopy exact sequence, say induced by a k-rational point $x : \mathbf{Spec}(k) \to X$, the outer Galois representation $\rho_{\text{out}} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Out}(\widehat{\pi}_1(X_{\overline{k}}, \overline{x}))$ lifts to a Galois representation $\rho_x : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{k}}, \overline{x}))$. We will show that this Galois representation recovers many of the familiar Galois representations appearing in algebraic geometry: for instance if $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, \infty\}$ the resulting Galois representation is given by the cyclotomic character, and if X is an Abelian variety over $\mathbb{Q}$ then the resulting Galois representation is given by the usual Galois action on the Tate module $\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \overline{x}) \simeq T(X) = \prod_\ell T_\ell(X)$.

One example which will appear repeatedly throughout the paper is the étale fundamental group of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ and its base change $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$. By comparison with the complex analytic space $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$ we have an isomorphism $\widehat{\pi}_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}, \overline{x}) \simeq \widehat{F}_2$ where $\widehat{F}_2$ is the profinite completion of the free group $F_2 = \langle x, y \rangle = \langle x, y, z \mid xyz = 1 \rangle$ where x, y, and z are closed loops around 0, 1, and ∞ respectively. We have a short exact sequence of étale fundamental groups

$$0 \to \widehat{F}_2 \to \widehat{\pi}_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}, \overline{x}) \to \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 0$$

The salience of the étale fundamental group of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ to our understanding of the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ begins with Belyi's three point theorem, that smooth proper irreducible curves over number fields are precisely those curves which, geometrically, are finite branched covers of $\mathbb{P}^1$ ramified over at most three points, which we may take to be $\{0, 1, \infty\}$ up to Möbius transformation. More precisely:

Theorem 0.1. *(Belyi) Let X be an integral proper normal curve over an algebraically closed field k of characteristic 0. Then X is defined over $\overline{\mathbb{Q}}$, hence over some number field, precisely if there exists a finite morphism $\beta : X \to \mathbb{P}^1_k$ unramified outside $\{0, 1, \infty\}$.*

We call such a morphism $\beta : X \to \mathbb{P}^1_k$ a Belyi morphism which presents the curve X. Among other things, this gives a way of drawing curves over number fields as bipartite graphs embedded in topological surfaces by lifting the standard interval $(0, 1)$ in $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$, called dessins d'enfants by Grothendieck. Specifically by Belyi's theorem a compact Riemann surface X is defined over $\overline{\mathbb{Q}}$ precisely if it can be realized as a branched cover $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ of the Riemann sphere ramified over $\{0, 1, \infty\}$. In this case we can use the path lifting properties of topological covering spaces to lift the interval $[0, 1]$ to X so as to obtain a bipartite graph whose vertices are colored according to the preimages $\beta^{-1}(0)$ and $\beta^{-1}(1)$. Of course there is no reason to prefer lifting the standard interval $(0, 1)$ as opposed to the intervals $(1, \infty)$ or $(\infty, 0)$, that is the image of $(0, 1)$ under automorphisms of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ permuting the points $\{0, 1, \infty\}$. Consequently given the automorphisms $\theta(t) = 1 - t$

and $\omega(t) = \frac{1}{1-t}$ of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$ we will interpret the compositions $\beta \circ \theta, \beta \circ \omega^{-1} : X \to \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$ in terms of combinatorial operations on the associated dessins: here the dessin associated to $\beta \circ \theta : X \to \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$ is the above dessin with the bipartite coloring switched, whereas the dessin associated to $\beta \circ \omega^{-1} : X \to \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$ amounts to taking the dual graph of the dessin.

Indeed one should view such bipartite graphs as presentations of the respective curves over number fields, in which case certain properties of the Belyi morphism and hence the curve can be deduced from certain combinatorial properties of the associated dessin: for instance the monodromy group can be computed in terms of edge permutations on the dessin, and in good cases this is enough to separate Galois orbits of the dessin and hence of the associated curves.

As the curves involved are defined over number fields there is a canonical action of the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on such curves which descends to an action on dessins. On the other hand since such curves over number fields can be realized as finite covers of $\mathbb{P}^1_k \setminus \{0,1,\infty\}$ there is a second action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Belyi functions and hence on dessins by geometric monodromy. More precisely, given the above étale homotopy exact sequence we obtain an outer Galois representation $\rho_{\text{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F}_2)$; since such dessins are represented by conjugacy classes of open subgroups in $\widehat{F}_2$, the outer Galois representation yields an action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on such conjugacy classes. We can represent this situation as a morphism of short exact sequences of profinite groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{F}_2 & \longrightarrow & \widehat{\pi}_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}, \overrightarrow{v}) & \underset{s_{\overrightarrow{v}}}{\leftrightarrows} & \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \overset{\rho_{\overrightarrow{v}}}{\swarrow} & \downarrow \rho_{\mathbf{out}} & & \\
0 & \longrightarrow & \mathbf{Inn}(\widehat{F}_2) & \longrightarrow & \mathbf{Aut}(\widehat{F}_2) & \longrightarrow & \mathbf{Out}(\widehat{F}_2) & \longrightarrow & 0
\end{array}$$

We are specifically interested in the outer Galois representation $\rho_{\mathbf{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F}_2)$ due to Belyi's theorem on the outer Galois action:

Theorem 0.2. *(Belyi) The outer Galois representation $\rho_{\mathbf{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F}_2)$ is faithful. In particular $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a profinite subgroup of the profinite outer automorphism group $\mathbf{Out}(\widehat{F}_2)$.*

Given a choice of section $s : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\pi}_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}, \overline{x})$ of the above étale homotopy exact sequence, the outer Galois representation $\rho_{\mathbf{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F}_2)$ lifts to a Galois representation $\rho : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{F}_2)$ which is again faithful by Belyi's theorem. We can do this in two ways, both of which we will show: the more elementary approach involves choosing representatives of each element $\gamma \in \mathbf{Out}(\widehat{F}_2)$ by hand, while the more advanced approach involves constructing a section of the étale homotopy short exact sequence so as to lift the entire outer Galois representation; the desired section $s_{\overrightarrow{v}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\pi}_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}, \overrightarrow{v})$ can be constructed given a choice of $\mathbb{Q}$-rational basepoint, or more generally a choice of $\mathbb{Q}$-rational tangential basepoint of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$. The main theorem we will show in this paper reveals an explicit calculation of this Galois representation and certain relations which express the compatibility of this Galois representation with the above recoloring and dualizing operations on dessins:

Theorem 0.3. *Each $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathbf{Aut}(\widehat{\pi}_1(X, \vec{01})) \simeq \mathbf{Aut}(\widehat{F}_2)$ is determined by parameters $(\chi(\sigma), \mathfrak{f}_\sigma) \in \widehat{\mathbb{Z}}^\times \times [\widehat{F}_2, \widehat{F}_2]$ where $\chi(\sigma) \in \widehat{\mathbb{Z}}^\times$ is the cyclotomic character and $\mathfrak{f}_\sigma \in [\widehat{F}_2, \widehat{F}_2]$ is the non-Abelian Galois symbol in the derived subgroup of $\widehat{F}_2$, acting on $\widehat{\pi}_1(X, \vec{01}) \simeq \widehat{F}_2 = \langle x, y \rangle^\wedge$ as*

$$\begin{cases} \sigma \cdot x = x^{\chi(\sigma)} \\ \sigma \cdot y = \mathfrak{f}_\sigma^{-1} y^{\chi(\sigma)} \mathfrak{f}_\sigma \end{cases}$$

Moreover the parameters $(\chi(\sigma), \mathfrak{f}_\sigma) \in \widehat{\mathbb{Z}}^\times \times \widehat{F}_2'$ *satisfy the profinite* 2*-cycle and* 3*-cycle relations in* $\widehat{F}_2$

$$\begin{cases} \mathfrak{f}_\sigma(y,x)\mathfrak{f}_\sigma(x,y) = 1 & (i) \\ \mathfrak{f}_\sigma(z,x)z^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(y,z)y^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(x,y)x^{\frac{\chi(\sigma)-1}{2}} = 1 & (ii) \end{cases}$$

These are the relations which define the coarse Grothendieck-Teichmüller group $\widehat{\mathbf{GT}}_0$ as a subgroup of $\mathbf{Out}(\widehat{F}_2)$ containing $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, through which the above outer Galois representation factors. Here $\mathfrak{f}_\sigma \in [\widehat{F}_2, \widehat{F}_2]$ is the Galois parameter describing the action of σ on the standard path p from 0 to 1, where for generators $u, v \in \widehat{F}_2$ the parameter $\mathfrak{f}_\sigma(u,v) \in [\widehat{F}_2, \widehat{F}_2]$ is the image of $\mathfrak{f}_\sigma \in [\widehat{F}_2, \widehat{F}_2]$ under the canonical isomorphism $\widehat{F}_2 \to \widehat{F}_2$ sending the distinguished generators $x, y \in \widehat{F}_2$ to u and v respectively. The above relations are induced by the underlying geometry of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, namely the action of the automorphisms θ and ω on the straight path p from 0 to 1 in $\mathbb{P}^1_{\mathbb{C}}$ satisfying the geometric relations $\theta(p)p = 1$ and $\omega^2(p)\omega(p)p = 1$. These relations are the source of the above profinite relations, which express the $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariance of the automorphisms θ and ω acting on dessins.

We will conclude by discussion how this situation can be generalized to that of Galois representations in étale fundamental groups of moduli spaces of curves over $\mathbb{Q}$, and how this gives rise to further relations on the above (outer) Galois representation. In particular we will discuss the additional relation that defines the Grothendieck-Teichmüller group $\widehat{\mathbf{GT}}$, and we will give a brief survey of the evidence that this is isomorphic to the absolute Galois group.

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1 Fundamental Groups

We will begin by reviewing some aspects of the fundamental group of topological spaces which we will export to the case of schemes. Specifically, we will discuss the relationship between the fundamental group of a topological space and its covering spaces, and the notions of monodromy and universal covers that come with it. For this we refer to [Hatcher] and [Dieck] for the basic theory.

It is well known that for X a sufficiently nice topological space and for $x \in X$ a basepoint, the fundamental group $\pi_1(X, x)$ of X at x is the set of homotopy classes of loops in X based at x, regarded as a group with respect to composition of paths; more generally the fundamental groupoid $\Pi_1(X)$ is the set of homotopy classes of paths in X, regarded as a groupoid with vertex groups $\pi_1(X, x)$ indexed by points $x \in X$ and edge torsors $\pi_1(X; x, y)$ indexed by pairs of points $x, y \in X$. This is a particularly workable definition since paths are easily visualized, with the only difficulty coming from determining which homotopy class each path belongs to. Moreover by using the path lifting and homotopy lifting properties of covering spaces, we can derive all the usual covering space properties of the fundamental group.

Unfortunately this approach does not generalize well to the case of schemes, where we no longer have an appropriate notion of path and homotopy. For this reason we will view covering spaces and their monodromy as primary notions, whereas we will view paths and homotopies as secondary notions. For instance we will view paths and homotopies as natural isomorphisms of fiber functors, only viewing these as geometric paths and homotopies when permissible.

1.1 Covering Spaces

Recall that a bundle over a topological space X is a surjective continuous map $p : E \to X$, where for each $x \in X$ the preimage $F_x = p^{-1}(x)$ is called the fiber of p at x. Of course this is exceedingly general, as such a bundle can be extremely nontrivial even locally, and its fibers can vary wildly over the base space X.

On the other hand certain bundles, most notably trivial bundles, admit a simple description: for F a fixed topological space the trivial bundle $p : X \times F \to X$ with fiber F over each $x \in X$ is the product bundle given by the continuous projection $p(x, y) = x$. In this context the notion of a covering space is easy to state: it is a fiber bundle which locally looks like the trivial bundle whose fibers are discrete topological spaces.

Definition 1.1. *Let X be a topological space. A covering space of X is a (surjective) continuous map $p : E \to X$ which is locally trivial with discrete fibers, that is if for each $x \in X$ there exists an open neighborhood U of x such that the pullback bundle $p|_U : p^{-1}(U) \to U$ is isomorphic to the trivial bundle with discrete fiber $p^{-1}(x)$.*

$$\begin{array}{ccccc} U \times p^{-1}(x) & \xrightarrow{\sim} & p^{-1}(U) & \longrightarrow & E \\ & \searrow & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ & & U & \hookrightarrow & X \end{array}$$

We write $\mathbf{Cov}_{/X}$ for the category of covering spaces over X, that is the full subcategory of the slice category $\mathbf{Top}_{/X}$ of topological spaces over X on those continuous maps $E \to X$ which are covering spaces. We will allow for covering spaces with empty fibers, but we will typically assume that $p : E \to X$ is surjective so that the fibers are nonempty. Of course distinct points $x, y \in X$ may have non-isomorphic fibers, but any path $\gamma : x \to y$ in X induces an isomorphism of fibers $\gamma : p^{-1}(x) \xrightarrow{\sim} p^{-1}(y)$.

Crucially, taking fibers over a point $x \in X$ is functorial in the sense that each morphism $f : p_0 \to p_1$ of covering spaces $p_0 : E_0 \to X$ and $p_1 : E_1 \to X$ induces a morphism of fibers $f_* : p_0^{-1}(x) \to p_1^{-1}(x)$. Indeed each point $x \in X$ defines a fiber functor for $\mathbf{Cov}_{/X}$:

Definition 1.2. *Let X be a topological space and let $x \in X$ be a basepoint. The fiber functor associated to x is the functor $F_x : \mathbf{Cov}_{/X} \to \mathbf{Set}$ sending each covering space $p : E \to X$ to the fiber $p^{-1}(x)$ and sending each morphism $f : p_0 \to p_1$ of covering spaces $p_0 : E_0 \to X$ and $p_1 : E_1 \to X$ to the induced morphism $f_* : p_0^{-1}(x) \to p_1^{-1}(x)$.*

It is easy to see that each path $\gamma : x \to y$ in X yields a natural isomorphism $F_x \simeq F_y$, since all of the induced morphisms yield a commutative diagram

$$\begin{array}{ccc} p_0^{-1}(x) & \xrightarrow{f_*} & p_1^{-1}(x) \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\ p_0^{-1}(y) & \xrightarrow{f_*} & p_1^{-1}(y) \end{array}$$

Moreover for $\delta : x \to y$ another path homotopic to $\gamma : x \to y$ the natural isomorphisms $F_x \simeq F_y$ induced by γ and δ are equal. For this reason we will refer to certain functors $F : \mathbf{Cov}_{/X} \to \mathbf{Set}$ as points of X, even if such a functor is not induced by a point $x \in X$; for instance if X is not compact, so called points at infinity will give rise to perfectly reasonable fiber functors $F : \mathbf{Cov}_{/X} \to \mathbf{Set}$, even though such points do not belong to X.

Lemma 1.1. *Let $p : E \to X$ be a covering space, and consider the diagonal morphism $\Delta : E \to E \times_X E$. Then the image $\Delta(E)$ is both open and closed in $E \times_X E$.*

Proof. On one hand let $x \in E$ and let U_x be an open neighborhood of x mapping homeomorphically onto its image under $p : E \to X$. Then $V_x = \Delta \cap (U_x \times U_x)$ is an open neighborhood of $(x, x) \in E \times_X E$ contained in Δ, so it follows that Δ is open in $E \times_X E$. On the other hand let $x \neq y \in E$ with $p(x) \neq p(y)$, and let U_x and U_y be open neighborhoods of x and y respectively mapping homeomorphically onto an open neighborhood U of $p(x) = p(y)$. Since $x \neq y$ we have $U_x \cap U_y = \emptyset$ hence $\Delta \cap (U_x \times U_y)$ is an open neighborhood of $(x, y) \in E \times_X E$ disjoint from Δ, so it follows that Δ is closed in $E \times_X E$. □

From this we can immediately deduce the unique lifting property of covering spaces: if two lifts of a continuous map $f : Y \to X$ along a covering space $p : E \to X$ are equal at some point, then the lifts are equal. More precisely:

Lemma 1.2. *(Unique Lifting Property) Let $p : E \to X$ be a covering space, let Y be a connected space, let $f : Y \to X$ be a continuous function, and let $\widetilde{f}_0, \widetilde{f}_1 : Y \to E$ be continuous lifts of f so that $f = \widetilde{f}_0 \circ p = \widetilde{f}_1 \circ p$. If $\widetilde{f}_0(y) = \widetilde{f}_1(y)$ for some $y \in Y$, then $\widetilde{f}_0 = \widetilde{f}_1$.*

$$\begin{array}{ccc} & \overset{f_1}{\nearrow} \; \overset{f_0}{\nearrow} & E \\ & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

Proof. Consider the canonical map $(\widetilde{f}_0, \widetilde{f}_1) : Y \to E \times_X E$, and the diagonal $\Delta(E) \hookrightarrow E \times_X E$. Then $\Delta(E)$ is both open and closed in $E \times_X E$, so by continuity of $(\widetilde{f}_0, \widetilde{f}_1)$ the pullback $(\widetilde{f}_0, \widetilde{f}_1)^{-1}(\Delta(E))$ is both open and closed in Y. But by assumption $(\widetilde{f}_0, \widetilde{f}_1)^{-1}(\Delta(E))$ is nonempty, and since Y is connected it follows that $(\widetilde{f}_0, \widetilde{f}_1)^{-1}(\Delta(E)) = Y$ But then it follows that $\widetilde{f}_0 = \widetilde{f}_1$ as claimed. □

The main lifting property on covering spaces we will employ is the Hurewicz lifting property [1], that is the right lifting property against continuous functions of the form $(1_Y, 0) : Y \to Y \times [0, 1]$ with Y a connected locally path connected space. For brevity we will only state the lifting property, focusing more on the unique path lifting and unique homotopy lifting properties that arise as a consequence:

[1] In other words covering spaces are Hurewicz fibrations: whereas fiber bundles are those surjective continuous maps $E \to X$ with each fiber homeomorphic to a fixed topological space F, Hurewicz fibrations are those surjective continuous maps $E \to X$ with each fiber homotopy equivalent to a fixed topological space F.

Theorem 1.3. *(Homotopy Lifting Property) let $p : E \to X$ be a covering space, let Y be a connected locally path connected space, and let $f : Y \to E$ and $\eta : Y \times [0,1] \to X$ be continuous maps. Then there is a unique lift $\widetilde{\eta} : Y \times [0,1] \to E$ so that $\eta = \widetilde{\eta} \circ p$ yielding the commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f} & E \\ {\scriptstyle (1_Y,0)}\downarrow & \overset{\widetilde{\eta}}{\nearrow} & \downarrow{\scriptstyle p} \\ Y \times [0,1] & \xrightarrow{\eta} & X \end{array}$$

The special cases we are interested in are the homotopy lifting property where $Y = [0,1]$ and $Y = *$, in which case the relevant lifting problem is against morphisms of the form $* \to [0,1]$ and $[0,1] \to [0,1] \times [0,1]$. In the first case such a morphism represents the point 0 in $[0,1]$, in which case the lifting property is the path lifting property:

Example 1.1. *Let $p : E \to X$ be a covering space, let $x \in X$ be a point, let $\gamma \in \pi_1(X,x)$ be a path, and let $\widetilde{x} \in E$ be a lift of x, that is $\widetilde{x} \in p^{-1}(x)$. Then there exists a unique path $\widetilde{\gamma} \in \pi_1(E, \widetilde{x})$ lifting γ so that $\gamma = \widetilde{\gamma} \circ p$. Indeed this follows by the homotopy lifting property for the following diagram:*

$$\begin{array}{ccc} * & \xrightarrow{\widetilde{x}} & E \\ {\scriptstyle 0}\downarrow & \overset{\widetilde{\gamma}}{\nearrow} & \downarrow{\scriptstyle p} \\ [0,1] & \xrightarrow{\gamma} & X \end{array}$$

In the second case such a morphism represents the union $[0,1] \times \{0\}$ with $\{0,1\} \times [0,1]$ which is isomorphic to $[0,1]$, in which case the morphism $[0,1] \to [0,1] \times [0,1]$ represents the identity morphism $[0,1] \to [0,1]$ in the first factor and the point 0 in the second factor, in which case the lifting property is the homotopy lifting property:

Example 1.2. *Let $p : E \to X$ be a covering space and let $\eta : \gamma \to \delta$ be a homotopy of paths $\gamma, \delta \in \Pi_1(X)$. Then for each lift $\widetilde{\gamma}, \widetilde{\delta} \in \Pi_1(E)$ of γ and δ to E there exists a unique homotopy $\widetilde{\eta} : \widetilde{\gamma} \to \widetilde{\delta}$ lifting the homotopy η so that $\eta = \widetilde{\eta} \circ p$. Indeed this follows by the homotopy lifting property of the following diagram:*

$$\begin{array}{ccc} [0,1] & \xrightarrow{(\gamma,\delta)} & E \\ {\scriptstyle (1_I,0)}\downarrow & \overset{\widetilde{\eta}}{\nearrow} & \downarrow{\scriptstyle p} \\ [0,1] \times [0,1] & \xrightarrow{\eta} & X \end{array}$$

Both the unique path lifting property and the unique homotopy lifting property give the following recipe for path lifting: given a covering space $p : E \to X$ with a point $x \in X$, each path $\gamma \in \pi_1(X,x)$ lifts to a path $\widetilde{\gamma}$ in E between two points in the fiber F_x, and this lifting is well-defined since any other path δ in the homotopy class of γ lifts to a path $\widetilde{\delta}$ which is uniquely homotopic to $\widetilde{\gamma}$. This is precisely the lifting process that is used to produce the monodromy representation of the fundamental group in the typical fiber of a covering space, which we turn to next.

1.2 Fundamental Groups

in view of the lifting properties of covering spaces there is a canonical way in which the fundamental groupoid $\Pi_1(X)$ acts on the fibers of a covering space $p : E \to X$. Consequently we can assign to each covering space E of X a permutation representation of $\Pi_1(X)$ on the set F_x called its monodromy, and this assignment is an equivalence of categories in that for any such permutation representation we can construct a covering space with the same monodromy.

Definition 1.3. *Let X be a locally path connected space with fundamental groupoid $\Pi_1(X)$, and let $p : E \to X$ be a covering space of X. The monodromy functor $\mathbf{Fib}_E : \Pi_1(X) \to \mathbf{Set}$ is the functor sending each point $x \in X$ to the fiber $p^{-1}(\{x\})$, and sending each homotopy class of paths $\gamma : x \to y$ to the induced function $\gamma^* : p^{-1}(x) \to p^{-1}(y)$ on fibers.*

Indeed for $f : E_0 \to E_1$ be a morphism of covering spaces of a locally simply connected space X it follows from the definition that f induces a natural transformation of monodromy functors $\mathbf{Fib}_f : \mathbf{Fib}_{E_0} \to \mathbf{Fib}_{E_1}$.

There are two additional ways in which this construction is functorial. On one hand for $f : X \to Y$ a continuous map of topological spaces with $x \in X$ and $y = f(x) \in Y$ the induced morphism $f_* : \pi_1(X, y) \to \pi_1(Y, x)$ yields a functor $f^* : \pi_1(Y, y)\mathbf{Set} \to \pi_1(X, x)\mathbf{Set}$ which commutes with the pullback functor $f^* : \mathbf{Cov}_{/Y} \to \mathbf{Cov}_{/X}$ yielding a commutative diagram

$$\begin{array}{ccc} \mathbf{Cov}_{/Y} & \xrightarrow{\mathbf{Fib}_x} & \pi_1(Y,y)\mathbf{Set} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbf{Cov}_{/X} & \xrightarrow{\mathbf{Fib}_y} & \pi_1(X,x)\mathbf{Set} \end{array}$$

On the other hand for $x, y \in X$ two geometric points of a connected topological space X we obtain a natural isomorphism of fiber functors $\mathbf{Fib}_x \simeq F_y : \mathbf{Cov}_{/X} \to \mathbf{Set}$ which yields an isomorphism $\pi_1(X, x) \simeq \pi_1(X, y)$, although this isomorphism is not canonical.

Definition 1.4. *Let X be a locally path connected semilocally simply connected topological space, and let $\rho : \Pi_1(X) \to \mathbf{Set}$ be a permutation representation. The reconstruction space $\mathbf{Rec}(\rho)$ is the disjoint union topological space*

$$\mathbf{Rec}(\rho) = \coprod_{x \in X} \rho(x)$$

with the topology generated by subsets of the form

$$V_{U,\widetilde{x}} = \{\rho(\gamma)(\widetilde{x}) \mid y \in U, \gamma \in \Pi_1(X; x, y)\} \subseteq \mathbf{Rec}(\rho)$$

for $x \in X$, $\widetilde{x} \in \rho(x)$, and U a path connected open subset of X containing x such that the induced morphism $\pi_1(U, x) \to \pi_1(X, x)$ is the zero morphism.

Lemma 1.4. *Let X be a locally path connected semilocally simply connected topological space, and let $\rho : \Pi_1(X) \to \mathbf{Set}$ be a permutation representation. Then the reconstruction space $\mathbf{Rec}(\rho)$ is a covering space of X.*

Proof. We must first check that $\mathbf{Rec}(\rho)$ is a topological space. Let $x \in X$; since X is semilocally simply connected there exists an open neighborhood $U_x \subseteq X$ of x such that $\pi_1(U_x, x) = 0$ so that each loop on U_x based at x is contractable. Now since X is locally path connected there exists a path connected open neighborhood $U \subseteq U_x$ of x such that every loop in U based at x is contractable. But then by definition for each $\widetilde{x} \in U$ we have $\widetilde{x} \in V_{U,x}$ and hence each $\widetilde{x} \in \rho(x)$ is contained in a basis open subset. By similar reasoning for each of the basis open subsets $V_{U,x}$ and $V_{V,y}$ there exists some basis open subset $V_{W,z}$ contained in $V_{U,x} \cap V_{V,y}$.

Now we must show that the canonical projection $p : \mathbf{Rec}(\rho) \to X$ is continuous. Let U be an open subset of X which we may assume to be simply connected since X is semilocally simply connected; then since for each $x, y \in U$ there exists a unique element $\Pi_1(U; x, y)$ we have a bijection

$$p^{-1}(U) \simeq \coprod_{\widetilde{x} \in \rho(x)} V_{U,\widetilde{x}} \simeq U \times \rho(x)$$

But then $p^{-1}(U)$ is a union of basic opens hence open, so it follows that p is continuous.

Finally we must show that $p : \mathbf{Rec}(\rho) \to X$ is a covering space, that is for all $x \in X$ there exists an open neighborhood U of x such that $p^{-1}(U) \simeq U \times \rho(x)$. But since X is semilocally simply connected we may assume that U has $\pi_1(U, x) = 0$ in which case $p^{-1}(U) = U \times \rho(x)$ by the above identification. The result follows. □

Indeed for $f : \rho \to \sigma$ a morphism of permutation representations $\rho, \sigma : \Pi_1(X) \to \mathbf{Set}$, we obtain a morphism of covering spaces $\mathbf{Rec}(f) : \mathbf{Rec}(\rho) \to \mathbf{Rec}(\sigma)$ which is continuous since for $V_{U,\widetilde{x}}$ a basic open of $E(\sigma)$ we have

$$\mathbf{Rec}(f)^{-1}(V_{U,\widetilde{x}}) = \coprod_{\widetilde{y} \in f^{-1}(\widetilde{x})} V_{U,\widetilde{x}}$$

which is a disjoint union of basic opens hence open. Consequently we obtain a functor $\mathbf{Rec} : \pi_1(X, x)\mathbf{Set} \to \mathbf{Cov}_{/X}$ which we claim to be an inverse to the monodromy functor $\mathbf{Fib} : \mathbf{Cov}_{/X} \to \pi_1(X, x)\mathbf{Set}$. This is the main theorem on covering spaces:

Theorem 1.5. *Let X be a locally path connected semilocally simply connected topological space. Then the monodromy functor $\mathbf{Fib} : \mathbf{Cov}_{/X} \to \Pi_1(X)\mathbf{Set}$ and the reconstruction functor $\mathbf{Rec} : \Pi_1(X)\mathbf{Set} \to \mathbf{Cov}_{/X}$ are inverse isomorphisms of categories.*

$$\mathbf{Cov}_{/X} \underset{\mathbf{Fib}}{\overset{\mathbf{Rec}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \Pi_1(X)\mathbf{Set}$$

Proof. On one hand let $\rho \in \Pi_1(X)\mathbf{Set}$ be a permutation representation. We must exhibit a natural isomorphism $\eta_\rho : \rho \xrightarrow{\sim} \mathbf{Fib}(\mathbf{Rec}(\rho))$. Since for all $x \in X$ we have $\mathbf{Fib}(\mathbf{Rec}(\rho))(x) = \rho(x)$, it suffices to show that for all $\gamma \in \Pi_1(X)$ we have $\mathbf{Fib}(\mathbf{Rec}(\rho))(\gamma) = \rho(\gamma)$. Now by path lifting there exists paths $\gamma_1, ..., \gamma_n \in \Pi_1(X)$ such that $\gamma = \gamma_n \circ ... \circ \gamma_1$ such that eachγ_i factors through an open subset U_i of X trivializing $\mathbf{Rec}(\rho)$. But then by trivialization we have $\mathbf{Fib}(\mathbf{Rec}(\rho))(\gamma_i) = \rho(\gamma_i)$, and hence by functoriality we have $\mathbf{Fib}(\mathbf{Rec}(\rho))(\gamma) = \rho(\gamma)$. But since the components of η_ρ are equalities it follows that the identity $\eta_\rho : \rho \xrightarrow{\sim} \mathbf{Fib}(\mathbf{Rec}(\rho))$ is a natural isomorphism.

On the other hand let $p : E \to X$ be a covering space. We must exhibit a natural isomorphism of covering spaces $\varepsilon_E : \mathbf{Rec}(\mathbf{Fib}(E)) \xrightarrow{\sim} E$. Since the underlying set of $\mathbf{Rec}(\mathbf{Fib}(E))$ is equal to E, it suffices to show that the identity morphism $\varepsilon_E : \mathbf{Rec}(\mathbf{Fib}(E)) \xrightarrow{\sim} E$ is a homeomorphism.

Now since X is locally path connected and semilocally simply connected we can check this locally: for $U \subseteq X$ an open subset and $x \in X$ a point such that the induced morphism $\pi_1(U, x) \to \pi_1(X, x)$ is the zero morphism, by definition the open subsets of E of the form $U \times \{\widetilde{x}\} \subseteq p^{-1}(U)$ form a basis for the topology on $\mathbf{Rec}(\mathbf{Fib}(E))$ and hence $\varepsilon_E : \mathbf{Rec}(\mathbf{Fib}(E)) \to E$ is a homeomorphism. But since the components of ε_E are equalities it follows that the identity $\varepsilon_E : \mathbf{Rec}(\mathbf{Fib}(E)) \xrightarrow{\sim} E$ is a natural isomorphism and the result follows. □

Theorem 1.6. *Let X be a path connected topological space, and let E be a covering space of X. Then E is path connected precisely if the monodromy $\mathbf{Fib}_E$ is a transitive action, and E is simply connected precisely if the monodromy $\mathbf{Fib}_E$ is a free action.*

Proof. By definition the fundamental groupoid $\Pi_1(E)$ is given by the Grothendieck construction for the monodromy functor $\mathbf{Fib}_E : \Pi_1(X) \to \mathbf{Set}$, that is the groupoid whose objects are pairs $(x, \widetilde{x})$ with $x \in X$ and $\widetilde{x} \in \mathbf{Fib}_E(x)$, and whose morphisms $\widetilde{\gamma} : (x, \widetilde{x}) \to (y, \widetilde{y})$ are homotopy classes of paths $\gamma : x \to y$ in $\Pi_1(X)$ such that $\mathbf{Fib}_E(\gamma)(\widetilde{x}) = \widetilde{y}$. But then the claim is immediate: if $\mathbf{Fib}_E$ is transitive then for each $x, y \in E$ there exists a path $\gamma : x \to y$ so that E is path connected, and if $\mathbf{Fib}_E$ is free then for each $x, y \in E$ there is at most one homotopy class of path $\gamma : x \to Y$ so that E is simply connected. □

The case of simply connected covers is particularly important: in this situation we single out the so called universal cover, that is the unique simply connected covering space through which every other simply connected covering space factors, provided it exists.

Definition 1.5. *Let X be a path connected locally path connected topological space. A covering space $\widetilde{X} \to X$ is called a universal cover if $\widetilde{X}$ is connected and simply connected, and every other simply connected covering space $E \to X$ factors through $\widetilde{X}$.*

$$\begin{array}{ccc} E & \longrightarrow & \widetilde{X} \\ & \searrow & \downarrow \\ & & X \end{array}$$

The immediate property of simply connected covering spaces is that they are unique up to isomorphism as covering spaces, since the homotopy lifting property yields mutually inverse liftings between the universal covering spaces. Moreover every path connected semilocally simply connected space has a universal cover which we can construct through the universal monodromy representation:

Theorem 1.7. *Let X be a path connected, and semilocally simply connected topological space. Then there exists a universal cover $\widetilde{X} \to X$ which is unique up to isomorphism.*

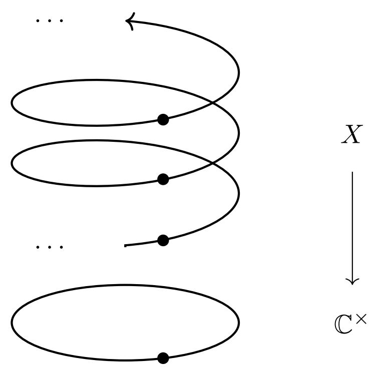
Proof. We have seen that a covering space $E \to X$ is simply connected precisely if its monodromy representation is free and transitive. Since X is locally path connected semilocally simply connected, every permutation representation of $\Pi_1(X)$ arises as the monodromy representation of some covering space. But for any point $x \in X$ the universal monodromy representation $F : \Pi_1(X) \to \mathbf{Set}$ given by $F(y) = \pi_1(X; x, y)$ is a free and transitive permutation representation of $\Pi_1(X)$ and hence represents a universal cover $\widetilde{X} \to X$. The result follows. □

In this situation a universal cover $\widetilde{X} \to X$ recovers the fundamental group of X as the group of automorphisms of the universal cover $\widetilde{X}$ fixing the base space:

$$\pi_1(X, x) = \mathbf{Aut}(\widetilde{X}/X)$$

In practice this gives us a way of computing the fundamental group of simple spaces by constructing the universal cover, in which case we can compute the fundamental groups of more complicated spaces using functoriality of the fundamental group and its interaction with constructions such as products, wedge sum, and open covers.

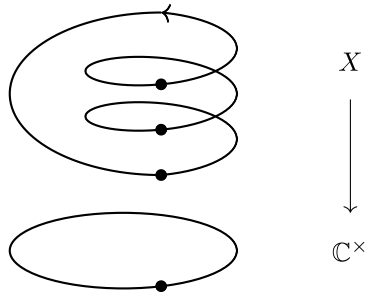
Example 1.3. *Let X be the Riemann surface associated to $f(z) = \log z$. This is constructed as follows: for U and U' two branches of $f(z)$ on domains $\mathbb{C} \setminus R$ and $\mathbb{C} \setminus R'$ with R and R' rays from the origin, we glue U and U' along the open subsets on which their respective values of $f(z)$ agree. Doing this for all possible choices of branch of $f(z)$ yields the desired Riemann surface: in this case X is homeomorphic to $\mathbb{C}$ and the covering space is given by the exponential morphism $\exp : \mathbb{C} \to \mathbb{C}^\times$; since X is simply connected and a $\mathbb{Z}$-principal cover it follows that $\Pi_1(\mathbb{C}^\times) \simeq \mathbb{Z}$.*



Indeed the monodromy around a simple closed positively oriented loop around 0 in $\mathbb{C}^\times$ increases the imaginary part of $\log(z)$ by $2\pi i$. The monodromy representation $\Pi_1(\mathbb{C}^\times) \to \mathbf{Fin}$ is up to choice of orientation the morphism $\mathbb{Z} \to \mathbf{Aut}(F_x)$ sending the generator $1 \in \mathbb{Z}$ to the infinite cyclic translation $(+1) \in \mathbf{Aut}(F_x)$ of the fiber F_x; this free and transitive and indeed $X \simeq \mathbb{C}$ is the universal cover of $\mathbb{C}^\times$ so that $\pi_1(\mathbb{C}^\times, x) \simeq \mathbb{Z}$.

In particular since $\mathbb{C}^\times$ deform retracts onto S^1 it follows that $\pi_1(S^1, x) \simeq \mathbb{Z}$ as expected. Indeed the (finite) permutation representations of $\mathbb{Z}$ are particularly simple, and we can easily describe the associated covering spaces:

Example 1.4. *Consider the Riemann surface X associated to $f(z) = \sqrt[n]{z}$. By a similar argument to the above this is a degree n cover of $\mathbb{C}^\times$ such that a simple closed loop around 0 in $\mathbb{C}^\times$ lifts to a path in X between the two points in the fiber of the base point of the loop. Indeed the monodromy representation $\Pi_1(\mathbb{C}^\times) \to \mathbf{Fin}$ is up to choice of orientation the morphism $\varphi : \mathbb{Z} \to S_n$ sending the generator $1 \in \mathbb{Z}$ to the cyclic permutation $(1, ..., n)$ so that the covering space X corresponds to the quotient $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ given by the unique conjugacy class of normal subgroup $n\mathbb{Z} \subseteq \mathbb{Z}$ of index n in $\mathbb{Z}$; this is transitive and indeed X is path connected.*



In general a permutation representation of $\mathbb{Z}$ is specified by a set F and a permutation $\sigma \in S_F$. By decomposing σ into cycles such a permutation representation is specified by a partition $F = \coprod_{i\in I} F_i$ and for each $i \in I$ a cycle $\sigma_i \in S_{F_i}$ of order $\#F_i$. Indeed any covering space of $\mathbb{C}^\times$ is isomorphic to a coproduct of (possibly infinite) cyclic Galois cover of $\mathbb{C}^\times$, and we obtain an equivalence of categories $\mathbf{Cov}_{\mathbb{C}^\times} \simeq \mathbb{Z}\mathbf{Set}$ as expected.

For instance with X as above consider the coproduct $X \amalg X \to \mathbb{C}^\times$; then the associated monodromy representation $\Pi_1(\mathbb{C}^\times) \to \mathbf{Fin}$ is given by the morphisms $\mathbb{Z} \to S_{2n}$ sending the generator $1 \in \mathbb{Z}$ to the cyclic permutations $(1...n)$ and $(n+1...2n)$ respectively; this is not transitive and indeed $X \amalg X$ is not path connected.

Example 1.5. *Let X and Y be path connected topological spaces, and let $x_0 \in X_0$ and $x_1 \in X_1$. The wedge sum $X_0 \vee X_1$ is the quotient $X_0 \vee X_1 = (X_0 \amalg X_1)/\sim$ by the relation identifying $x_0 \sim x_1$ and nothing else. Now suppose that we have presentations $\pi_1(X_0, x_0) = \langle \alpha_1, ..., \alpha_n \mid u_1, ..., u_p\rangle$ and $\pi_1(X_1, x_1) = \langle \beta_1, ..., \beta_m \mid v_1, ..., v_q\rangle$. Then $\pi_1(X_0 \vee X_1, x)$ is given as the free product $\pi_1(X_0, x_0) * \pi_1(X_1, x_1)$, that is we have the presentation*

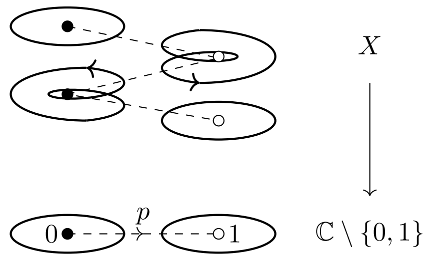
$$\pi_1(X_0 \vee X_1, x) = \langle \alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m \mid u_1, ..., u_p, v_1, ..., v_q\rangle$$

For instance consider the space $X = \mathbb{C} \setminus \{x_0, ..., x_n\}$ which deform retracts onto the wedge sum $(S^1)^{\vee n}$ so that X has fundamental group $\pi_1(X, x) \simeq F_n$ generated by loops $\gamma_0, ..., \gamma_n$ around $x_0, ..., x_n$ respectively with $\gamma_0 ... \gamma_n = 1$. Then we have an equivalence of categories $\mathbf{Cov}_{/X} \simeq F_n\mathbf{Set}$, and in general a permutation representation of F_n is specified by a set F and permutations $\sigma_0, ..., \sigma_n \in S_F$ with $\sigma_0 ... \sigma_n = 1$. In particular for $n > 1$ a permutation representation of F_n is equivalently specified by a set F and permutations $\sigma_0, ..., \sigma_{n-1} \in S_F$.

From this it immediately follows that every finitely presented group G can be realized as the monodromy group of a connected covering space X of $\mathbb{C} \setminus \{x_0, ..., x_n\}$ for some $n \geq 1$. Indeed such a group G is a quotient of some free group F_n by a normal subgroup N, and the coset permutation representation $F_n \to F_n/N$ yields the desired covering space by reconstruction.

As a special case consider the space $\mathbb{C} \setminus \{0, 1\}$ which is easily seen to deform retract onto the wedge sum $S^1 \vee S^1$ with fundamental group $\pi_1(\mathbb{C} \setminus \{0, 1\}, x_0) \simeq F_2$ generated by two loops x and y around 0 and 1 respectively. In general a permutation representation of F_2 is specified by a set F and two permutations $\sigma_0, \sigma_1 \in S_F$, and we obtain an equivalence of categories $\mathbf{Cov}_{\mathbb{C}\setminus\{0,1\}} \simeq F_2\mathbf{Set}$ as expected.

For instance let $X \to \mathbb{C} \setminus \{0, 1\}$ be a covering space of degree 3, and suppose that x and y act on the fiber $F = \{1, 2, 3\}$ near 0 by the permutations (12) and (23) respectively. The resulting monodromy representation $\pi_1(\mathbb{C} \setminus \{0, 1\}, x_0) \to S_3$ completely determines the covering space X: we have ramification of order 2 above 0 and 1 respectively, and the straight path p from 0 to 1 lifts to X as indicated.



As we will see, this picture cleanly summarizes the combinatorial theory of Grothendieck's dessins. Namely every integral proper normal curve over a number field can be presented by a finite cover $\beta : X \to \mathbb{C} \setminus \{0,1\}$ which is specified by the permutations σ_0, σ_1 of the fiber $F = \beta^{-1}(I)$ where $I = (0,1)$ is the standard open interval from 0 to 1. Equivalently given the ramification indices of points above 0 and 1 such a finite cover is equivalently specified by lifts of the path p from 0 to 1.

Example 1.6. *Let X_0 and X_1 be path connected topological spaces, and let $x_0 \in X_0$ and $x_1 \in X_1$. Suppose that we have presentations $\pi_1(X_0, x_0) = \langle \alpha_1, ..., \alpha_n \mid u_1, ..., u_p \rangle$ and $\pi_1(X_1, x_1) = \langle \beta_1, ..., \beta_m \mid v_1, ..., v_q \rangle$. Then the product $\pi_1(X_0 \times X_1, x)$ is given as the product $\pi_1(X_0, x_0) \times \pi_1(X_1, x_1)$, that is we have the presentation*

$$\begin{aligned}\pi_1(X_0 \times X_1, x) = \langle \alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m \mid u_1, ..., u_p, v_1, ..., v_q \\ [\alpha_1, \beta_1] = ... = [\alpha_n, \beta_1] = ... = [\alpha_1, \beta_m] = ... = [\alpha_n, \beta_m] = 1 \rangle\end{aligned}$$

For instance let E be a compact topological surface of genus 1, that is a complex torus of dimension 1. Then E is homeomorphic to the product $S^1 \times S^1$ and hence its fundamental group has presentation $\pi_1(E, x) = \langle \alpha, \beta \mid [\alpha, \beta] = 1 \rangle$ which is isomorphic to the free Abelian group $\mathbb{Z}^2$. More generally let X be a complex torus of dimension g. Then X is homeomorphic to the product $X^{\times g} \simeq (S^1)^{\times 2g}$ and hence its fundamental group has presentation

$$\pi_1(X, x) = \langle \alpha_1, \beta_1, ..., \alpha_g, \beta_g \mid [\alpha_1, \beta_1] = ... = [\alpha_g, \beta_g] = 1 \rangle$$

which is isomorphic to the free Abelian group $\mathbb{Z}^{2g}$.

Example 1.7. *Let X be a (nice) topological space with an open cover $X = U \cap V$. By the Van-Kampen theorem we have a (homotopy) pushout of fundamental groupoids*

$$\begin{array}{ccc} \Pi_1(U \cap V) & \longrightarrow & \Pi_1(U) \\ \downarrow & \ulcorner & \downarrow \\ \Pi_1(V) & \longrightarrow & \Pi_1(X) \end{array}$$

This pushout is easily calculated in the following case: suppose U, V, and $U \cap V$ are nonempty path connected subspaces of X, let $x \in U \cap V$, and suppose their fundamental groups have presentations $\pi_1(U, x) = \langle \alpha_1, ..., \alpha_n \mid u_1, ..., u_p \rangle$, $\pi_1(V, x) = \langle \beta_1, ..., \beta_m \mid v_1, ..., v_q \rangle$, and $\pi_1(U \cap V, x) = \langle \gamma_1, ..., \gamma_l \mid w_1, ..., w_r \rangle$. Then $\pi_1(X, x)$ is given as the free product $\pi_1(U, v) *_{\pi_1(U \cap V, x)} \pi_1(V, x)$ with amalgamation, that is for $I : \pi_1(U \cap V, x) \to \pi_1(U, x)$ and $J : \pi_1(U \cap V, x) \to \pi_1(V, x)$ the induced morphisms we have the presentation

$$\pi_1(X, x) = \langle \alpha_1, ..., \alpha_n, \beta_1, ...\beta_m \mid u_1, ..., u_p, v_1, ..., v_q, I(w_1)J(w_1)^{-1}, ..., I(w_r)J(w_r)^{-1} \rangle$$

For instance let X be a topological surface of genus g with n punctures, and let $x \in X$. Then it is easily seen by induction on the genus and the number of punctures that $\pi_1(X,x)$ is isomorphic to the group $\Pi_{g,n}$ with presentation

$$\Pi_{g,n} = \langle \alpha_1, \beta_1, ..., \alpha_g, \beta_g, \gamma_1, ..., \gamma_n \mid [\alpha_1, \beta_1]...[\alpha_g, \beta_g]\gamma_1...\gamma_n = 1\rangle$$

We will see generalizations of all the above examples to the setting of étale fundamental groups, namely the étale fundamental groups of the punctured projective line, of Abelian varieties, and of punctured curves over algebraically closed field of arbitrary characteristic will each be isomorphic to appropriate completions of the above topological fundamental groups. Namely in characteristic 0 this will be a direct consequence of the Riemann existence theorem, while in positive characteristic this will follow from Grothendieck's specialization theorem and lifting to characteristic 0.

1.3 Homotopy Exact Sequence

We have seen that for $p : E \to X$ a covering space and $x \in X$ we have a monodromy action $\pi_1(X,x) \times \pi_0(F_x) \to \pi_0(F_x)$ given by path lifting. For $y \in F_x$ and $\pi_0(F_x, y)$ the pointed set of connected components of F_x, we can define a boundary map

$$\partial_y : \pi_1(X,x) \to \pi_0(F_x, y)$$

given by $\partial_y(\gamma) = \gamma \cdot y$ which is $\pi_1(X,x)$-equivariant, that is $\partial_y(\delta \circ \gamma) = \delta \cdot \partial_y(\gamma)$. In this case the morphism $\partial_x : \pi_1(X,x) \to \pi_0(F_x, y)$ is the boundary map for a long exact sequence of homotopy groups associated to a fibration $f : E \to X$.

Theorem 1.8. *Let $f : E \to X$ be a fibration, let $x \in X$, let $y \in F_x = f^{-1}(x)$, and let $i : F_x \hookrightarrow E$ be the canonical inclusion. Then the morphism $\partial_y : \pi_1(X,x) \to \pi_0(F_x, y)$ yields an exact sequence*

$$\ldots \to \pi_1(F_x, y) \xrightarrow{i_*} \pi_1(E, y) \xrightarrow{f_*} \pi_1(X, x) \xrightarrow{\partial_y} \pi_0(F_x, y) \xrightarrow{i_*} \pi_0(E, y) \xrightarrow{f_*} \pi_0(X, x) \to 0$$

In particular if $\pi_2(X,x) \simeq 0$ or $\pi_1(F_x, y)$ has trivial center, then we have an exact sequence

$$0 \to \pi_1(F_x, y) \xrightarrow{i_*} \pi_1(E, y) \xrightarrow{f_*} \pi_1(X, x) \xrightarrow{\partial_y} \pi_0(F_x, y) \xrightarrow{i_*} \pi_0(E, y) \xrightarrow{f_*} \pi_0(X, x) \to 0$$

Now let $f : E \to B$ be a locally trivial fibration, say a covering space. Fix a basepoint $x \in B$, fix a point $y \in E$ over x, and consider the fiber F_x containing the point y. Then the fiber sequence $F_x \to E \to B$ induces exact sequence of fundamental groups

$$\pi_1(F_x, y) \to \pi_1(E, y) \to \pi_1(B, x) \to 0$$

In particular this extends to a short exact sequence of fundamental groups if $\pi_2(B, x) = 0$ or $\pi_1(F_x, y)$ has trivial center

$$0 \to \pi_1(F_x, y) \to \pi_1(E, y) \to \pi_1(B, x) \to 0$$

Now for $\sigma \in \pi_1(B, x)$ we can consider the deformation of the fiber F_x along σ. Namely, for $\gamma \in \pi_1(F_x, y)$ we can define the outer monodromy representation as follows: choose a lift $\widetilde{\sigma} \in \pi_1(E, y)$ of $\sigma \in \pi_1(B, x)$ and define the outer action of σ on γ as

$$\rho_{\text{out}}(\sigma)(\gamma) = \widetilde{\sigma} \circ \gamma \circ \widetilde{\sigma}^{-1}$$

where γ is considered as an element of $\pi_1(E, y)$ with respect to the canonical inclusion $\pi_1(F_x, y) \hookrightarrow \pi_1(E, y)$. Of course the automorphism $\rho_{\text{out}}(\sigma) \in \mathbf{Aut}(\pi_1(F_x, y))$ depends on the choice of lift $\widetilde{\sigma}$, but any other lift is of the form $\alpha \circ \widetilde{\sigma}$ for $\alpha \in \pi_1(F_x, y)$ and hence this determines a well-defined element $\rho_{\text{out}}(\sigma) \in \mathbf{Aut}(\pi_1(F_x, y))/\mathbf{Inn}(\pi_1(F_x, y)) = \mathbf{Out}(\pi_1(F_x, y))$. This is easily seen to extend to a group homomorphism $\rho_{\text{out}} : \pi_1(B, x) \to \mathbf{Out}(\pi_1(F_x, y))$ which we call the outer monodromy representation of $f : E \to B$ at y over x.

Suppose now that we have chosen a section $s : B \to E$ of $f : E \to B$ so that we obtain a section $s : \pi_1(B, x) \to \pi_1(E, y)$ of the above exact sequence by functoriality. Then each $\sigma \in \pi_1(B, x)$ has a canonical lift $s(\sigma) \in \pi_1(E, y)$, and hence for $\gamma \in \pi_1(F_x, y)$ we can define the monodromy representation $\rho_s : \pi_1(B, x) \to \mathbf{Aut}(\pi_1(F_x, y))$ as follows:

$$\rho_s(\sigma)(\gamma) = s(\sigma) \circ \gamma \circ s(\sigma)^{-1}$$

Since the lift $s(\sigma)$ is explicitly chosen this defines a well-defined element $\rho(\sigma) \in \mathbf{Aut}(\pi_1(F_x, y))$. Since the section $s : \pi_1(B, x) \to \pi_1(E, y)$ is a group homomorphism it follows that this extends to a group homomorphism $\rho_s : \pi_1(B, x) \to \mathbf{Aut}(\pi_1(F_x, y))$ which we call the monodromy representation of $f : E \to B$ at y over x with respect to the section $s : B \to E$.

We can summarize this situation as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(F_x, y) & \longrightarrow & \pi_1(E, y) & \underset{\overset{s}{\longleftarrow}}{\longrightarrow} & \pi_1(B, x) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \overset{\rho_s}{\swarrow} & \downarrow \rho_{\text{out}} & & \\
0 & \longrightarrow & \mathbf{Inn}(\pi_1(F_x, y)) & \longrightarrow & \mathbf{Aut}(\pi_1(F_x, y)) & \longrightarrow & \mathbf{Out}(\pi_1(F_x, y)) & \longrightarrow & 0
\end{array}$$

This will be extremely important later, as this will be the basis for the construction of the (outer) monodromy representation of étale fundamental groups, and in particular the (outer) Galois representation on the étale fundamental group of an integral normal scheme over a field. Until then, the main difficulty we must solve in the case of étale fundamental groups is the absence of universal covers in the category of schemes: we cannot take universal covers of the underlying topological space as the Zariski topology is almost never semilocally simply connected, and in most cases universal covers are not algebraic. Instead we must approach this situation synthetically, which we turn to below.

2 Étale Fundamental Groups

As we have seen, the relationship between fundamental groups and covering spaces is particularly clear in the presence of universal covers: every other covering space is subordinate to the universal one, and every such cover can be identified with a permutation representation of the fundamental groupoid. Of course in the absence of universal covers the situation is somewhat more complicated: we can no longer construct the fundamental group as the automorphism group of the universal cover, and instead we must work with the inverse system of covers directly.

In the case of schemes, the lack of universal covers is typically because the would-be universal covering morphism is not a morphism of schemes. For instance consider the punctured affine line $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$: while the usual cyclic Galois covers $X \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ are finite étale covers of schemes, the universal cover is no longer algebraic; consequently the algebraic fundamental group of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ should be isomorphic to $\widehat{\mathbb{Z}} = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ as the inverse system of Galois groups of covers.

Indeed the profinite group $\widehat{\mathbb{Z}}$ has the behavior of a would-be fundamental group: its quotients by open normal subgroups of finite index are precisely the cyclic groups, and its finite permutation representations can be identified with finite covers of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. In general the étale fundamental group will have much the same behavior, in that for X a scheme and $\overline{x}$ a geometric basepoint the étale fundamental group $\widehat{\pi}_1(X,\overline{x})$ will be a profinite group whose finite permutation representations will correspond to schemes finite étale over X.

2.1 Galois Categories

As soon as we try to define the étale fundamental group $\widehat{\pi}_1(X,\overline{x})$ of a scheme X at a basepoint $\overline{x}$, we immediately run into a problem: it is not at all clear that such a profinite group should even exist; a priori there could exist some scheme X such that $\mathbf{FEt}_X$ is so unruly as to obstruct $\mathbf{FEt}_X$ from being equivalent to $G\mathbf{Fin}$ for any profinite group G. In order to fix this problem we will introduce the notion of a Galois category, which are defined in terms of certain universal properties which are easily verified for the category $\mathbf{FEt}_X$, but nevertheless characterize Galois categories as categories of finite permutation representations of profinite groups. For this we consult [SGA1] and [Stacks 52.3].

In order to understand the definition of Galois categories, we should first understand the important properties of categories of finite permutations representations of profinite groups, from which we can deduce certain properties characterizing these categories. Among other things, our characterization will include the existence of finite limits and finite colimits, am exact fiber functor $F : \mathcal{C} \to \mathbf{Fin}$ preserving these finite limits and finite colimits, and the ability to uniquely write each object as a finite coproduct of so called connected objects.

For $F : \mathcal{C} \to \mathbf{Set}$ a functor from a (small) category $\mathcal{C}$ there is a canonical injective map $\mathbf{Aut}(F) \hookrightarrow \prod_{X\in\mathcal{C}} \mathbf{Aut}(F(X))$ which is continuous with respect to the topology on $\mathbf{Aut}(F)$ induced by the product topology on $\prod_{X\in\mathcal{C}} \mathbf{Aut}(F(X))$ with each $\mathbf{Aut}(F(X))$ discrete; in particular each action morphism $\mathbf{Aut}(F) \times F(X) \to F(X)$ is continuous. This has the following universal property: for G a topological group and $G \to \mathbf{Aut}(F)$ a group homomorphism such that each action morphism $G \times F(X) \to F(X)$ is continuous, then $G \to \mathbf{Aut}(F)$ is continuous.

Lemma 2.1. *Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor from a small category $\mathcal{C}$. Then the canonical inclusion $\mathbf{Aut}(F) \to \prod_{X\in\mathcal{C}} \mathbf{Aut}(F(X))$ identifies $\mathbf{Aut}(F)$ with a closed subgroup of $\prod_{X\in\mathcal{C}} \mathbf{Aut}(F(X))$ with the product topology.*

Proof. Let $\xi = (\alpha_X) \in \prod_{X\in\mathcal{C}} \mathbf{Aut}(F(X))$ be an element not contained in $\mathbf{Aut}(F)$. There exists a morphism $f : X \to Y$ in $\mathcal{C}$ and $x \in F(X)$ such that $F(f)(\alpha_X(x)) \neq \alpha_Y(F(f)(x))$. Now consider the open neighborhoods $U = \{\beta_X \in \mathbf{Aut}(F(X)) \mid \alpha_X(x) = \beta_X(x)\}$ and $V = \{\gamma_Y \in \mathbf{Aut}(F(Y)) \mid \alpha_Y(F(f)(x)) = \gamma_Y(F(f)(x))\}$ of α_X and α_Y respectively. Then the product $U \times V \times \prod_{Z\neq X,Y\in\mathcal{C}} \mathbf{Aut}(F(Z))$ is an open neighborhood of ξ not meeting $\mathbf{Aut}(F)$, and hence $\mathbf{Aut}(F)$ is closed in $\prod_{X\in\mathcal{C}} \mathbf{Aut}(F(X))$, whence the claim. □

Corollary 2.2. *Let $F : \mathcal{C} \to \mathbf{Fin}$ be a functor from a small category $\mathcal{C}$. Then the canonical inclusion $\mathbf{Aut}(F) \hookrightarrow \prod_{X\in\mathcal{C}} \mathbf{Aut}(F(X))$ identifies $\mathbf{Aut}(F)$ with a closed subgroup of $\prod_{X\in\mathcal{C}} \mathbf{Aut}(F(X))$.*

Proof. Since each $F(X)$ is finite the topological group $\prod_{X\in\mathcal{C}} \mathbf{Aut}(F(X))$ is a profinite group, and the claim follows by the previous lemma. □

Recall that for G a topological group we can define its profinite completion $\widehat{G} = \varinjlim_N G/N$ as the limit taken over open normal subgroups $N \subseteq G$ of finite index, which is cofiltered since the finite intersection of open normal subgroups of finite index is again an open normal subgroup of finite index. By the universal property of limits we have a canonical morphism $i : G \to \widehat{G}$. This the profinite completion $\widehat{G}$ has the following universal property: for every morphism $f : G \to H$ with H profinite there exists a unique morphism $\widetilde{f} : \widehat{G} \to H$ yielding the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{i} & \widehat{G} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \widetilde{f}} \\ & & H \end{array}$$

The main theorem is that this group $\widehat{G}$ can be recovered from the forgetful functor $F : G\mathbf{Fin} \to \mathbf{Fin}$ alone by taking its profinite group of natural automorphisms.

Theorem 2.3. *Let G be a topological group and let $F : G\mathbf{Fin} \to \mathbf{Fin}$ be the forgetful functor from the category of finite permutation representations of G. then we have an isomorphism of topological groups $\mathbf{Aut}(F) \simeq \widehat{G}$.*

Proof. Since any morphism in $G\mathbf{Fin}$ commutes with the action of G, each $g \in G$ defines an automorphism of $F : G\mathbf{Fin} \to \mathbf{Fin}$, and hence we have canonical homomorphism $G \to \mathbf{Aut}(F)$ of topological groups. Now since every finite permutation representation of G is a finite coproduct of permutation representations of the form G/H_i for H_i an open subgroup of finite index with the canonical G-action. Then the intersection $U_i = \bigcap_{g \in G} gH_ig^{-1}$ is an open normal subgroup of finite index in G, and each U_i acts trivially on each G/H_i, and hence $U = \bigcap_{i \in I} U_i$ acts trivially on $F(X)$ so in particular the action $G \times F(X) \to F(X)$ is continuous. But then by the universal property of $\mathbf{Aut}(F)$ the morphism $G \to \mathbf{Aut}(F)$ is continuous.

Now by the universal property of the profinite completion $\widehat{G}$ there exists a continuous homomorphism $\widehat{G} \to \mathbf{Aut}(F)$ which is injective since G/U acts faithfully on itself. Now we must show that the homomorphism $\widehat{G} \to \mathbf{Aut}(F)$ has dense image, in which case it is surjective and the claim follows. To that end let $\gamma \in \mathbf{Aut}(F)$; with U as above each of the canonical morphisms $G/U \to G/H_i$ is surjective, so it suffices to show that there exists some $g \in G$ such that the action of γ and g on G/U coincide. Let $e \in G$ be the identity so that $\gamma(eU) = hU$ for some $h \in G$. Then for $k \in G$ the canonical right action $\rho_k : G/U \to G/U$ given by $\rho_k(gU) = gkU$ commutes with the action of γ, in which case $\gamma(kU) = \gamma(\rho_k(eU)) = \rho_k(\gamma(eU)) = \rho_k(hU) = hkU$, and hence for $g = h$ we have $\gamma(kU) = gkU$ and the result follows. □

Theorem 2.4. *Let G be a topological group and let $F : G\mathbf{Fin} \to \mathbf{Fin}$ be an exact functor. Then F is naturally isomorphic to the forgetful functor $F_G : G\mathbf{Fin} \to \mathbf{Fin}$.*

Proof. Let $X \in G\mathbf{Fin}$ be nonempty. Then $F(X)$ is nonempty since the pushout of the terminal morphism $G \to 1$ along itself is the terminal object $1 \in G\mathbf{Fin}$, and F preserves this pushout. Now let U be an open normal subgroup of finite index in G and note that $G/U \times G/U = \coprod_{gU \in G/U} G/U$. Then since F is exact we have $F(G/U) \times F(G/U) = \coprod_{gU \in G/U} F(G/U)$ and hence $F(G/U)$ has the same underlying set as G/U since $F(G/U)$ is nonempty. From this we conclude that the limit $\varprojlim_{U \subseteq G} F(G/U)$ is nonempty, taken over open normal subgroups of finite index in G.

Now let $\gamma = (\gamma_U) \in \varprojlim_{U \subseteq G} F(G/U)$ be an element of this limit. By the universal property of the above limit we can identify the fiber functor F with the colimit

$$F(X) \simeq \varinjlim_{U \subseteq G} \mathbf{Hom}(G/U, X) \simeq \mathbf{Hom}(\varprojlim_{U \subseteq G} G/U, X)$$

again taken over open normal subgroups of finite index in G, with the morphisms $f : G/U \to X$ corresponding to $f(eU) \in X = F_G(X)$. Then the element γ yields a morphism $\widetilde{\gamma} : F_G \to F$ where for $x \in X$ and $f : G/U \to X$ with $f(eU) = x$ we have $\widetilde{\gamma}_X(x) = F(f)(\gamma_U)$. We claim that $\widetilde{\gamma}$ induces a bijection $F_G(G/U) \xrightarrow{\sim} F(G/U)$ for every open normal subgroup U of finite index in G. Since the $F_G(G/U)$ and $F(G/U)$ have the same underlying sets it suffices to show that $\widetilde{\gamma}_{G/U}$ is surjective. To that end for $g \in G$ consider the stabilizer of $F(\rho_g) : F(G/U) \to F(G/U)$ which is equal to $F(\emptyset) = \emptyset$ if $g \notin U$ since ρ_x is transitive and F is exact; but then for $g_1, ..., g_n \in G/U$ a set of representatives for G/U the elements $F(\rho_{g_i})(\gamma_U)$ are pairwise distinct and $\widetilde{\gamma}_{G/U}(g_i U) = F(\rho_{g_i})(\gamma_U)$ in which case $\widetilde{\gamma}$ is surjective and the result follows. □

Indeed for $\mathcal{C}$ a category and $F : \mathcal{C} \to \mathbf{Fin}$ a functor the automorphism group $G = \mathbf{Aut}(F)$ is profinite and acts on each $F(X)$ for $X \in \mathcal{C}$; we obtain a functor $\mathcal{C} \to G\mathbf{Fin}$ sending each $X \in \mathcal{C}$ to the finite set $F(X)$ with the induced G-action, and sending each morphism $f : X \to Y$ in $\mathcal{C}$ to the induced morphism $F(X) \to F(Y)$ of finite permutation representations of G. In order to identify those categories $\mathcal{C}$ with fiber functors $F : \mathcal{C} \to \mathbf{Fin}$ such that the above functor is an equivalence of categories, we are led to the definition of a Galois category.

Definition 2.1. *A Galois category is a category $\mathcal{C}$ with fiber functor $F : \mathcal{C} \to \mathbf{Fin}$ such that*

(i) $\mathcal{C}$ has finite limits and finite colimits;
(ii) Every object in $\mathcal{C}$ is a (nontrivial) finite coproduct of connected objects in $\mathcal{C}$;
(iii) F is exact and reflects isomorphisms; in other words F preserves finite limits and finite colimits, and for $f : X \to Y$ a morphism in $\mathcal{C}$ with $F(f)$ an isomorphism we have that f is an isomorphism.

Here an object X in $\mathcal{C}$ is called connected if $\mathbf{Aut}(X)$ acts freely on X, or equivalently if X is not initial and X is irreducible with respect to finite coproducts: if $X \simeq U \amalg V$ then $X \simeq U$ or $X \simeq V$. It follows from the definition that F is faithful and that F preserves and reflects both monomorphisms and epimorphisms. Moreover any morphism $f : X \to Y$ in $\mathcal{C}$ with both X and Y connected is an epimorphisms, and for $f, g : X \to Y$ two morphisms with X connected such that $F(f)(x) = F(g)(x)$ for some $x \in X$, then $f = g$. This is left as an exercise.

Indeed it is easy to see that for G a profinite group the category $G\mathbf{Fin}$ is a Galois category with respect to the forgetful functor $F : G\mathbf{Fin} \to \mathbf{Fin}$: it has finite limits and finite colimits, every object $X \in G\mathbf{Fin}$ can be written as a (nontrivial) finite coproduct $X \simeq G/H_1 \amalg ... \amalg G/H_n$ with each G/H_i connected since the canonical action of G on each G/H_i is transitive. Moreover the forgetful functor $F : G\mathbf{Fin} \to \mathbf{Fin}$ is exact by the previous theorem, and reflects isomorphisms since isomorphisms of finite G-sets are precisely those isomorphisms which commute with the G-action. What is less easy to see is that these properties alone fix the structure of $G\mathbf{Fin}$. In order to show this we will construct $\mathbf{Aut}(F)$ as a limit of finite automorphism groups of so called Galois objects in $\mathcal{C}$, mimicking the isomorphism in the previous theorem.

We proceed as follows; note that for $X \in \mathcal{C}$ a connected object the automorphism group $\mathbf{Aut}(X)$ has order at most the cardinality of $F(X)$ since for $x \in F(X)$ and $g \in \mathbf{Aut}(X)$ we have $g(x) = x$ precisely if $g = 1_X$. In the case of strict equality we obtain the following definition:

Definition 2.2. *Let $\mathcal{C}$ be a Galois category with fiber functor $F : \mathcal{C} \to \mathbf{Fin}$. Then an object X in $\mathcal{C}$ is called Galois if the automorphism group $\mathbf{Aut}(X)$ has order precisely the cardinality of $F(X)$.*

Equivalently a object X of $\mathcal{C}$ is Galois if it is a connected object and $\mathbf{Aut}(X)$ acts transitively on X. This definition is intentionally similar to that given in Galois theory. For instance every object X of $\mathcal{C}$ a Galois category is subordinate to some Galois object Y in the sense that there exists a morphism $Y \to X$. In this case we write $\mathbf{Gal}(Y) = \mathbf{Aut}(Y)$ for the automorphism group and call this the Galois group of Y.

In this situation the profinite automorphism group $\mathbf{Aut}(F)$ is a universal Galois group for $\mathcal{C}$ in an appropriate sense, namely it acts transitively on all Galois objects of $\mathcal{C}$ at once. Since every object of $\mathcal{C}$ is dominated by a Galois object, we can say something slightly stronger:

Lemma 2.5. *Let $\mathcal{C}$ be a Galois category with fiber functor F, and let $G = \mathbf{Aut}(F)$ be the profinite automorphism group of F. Then for X a connected object in $\mathcal{C}$ the canonical action of G on $F(X)$ is transitive.*

Proof. Let $\mathcal{I}$ be the set of isomorphism classes of Galois objects in $\mathcal{C}$, and for each $i \in \mathcal{I}$ let X_i be a representative in $\mathcal{C}$. For each $i \in \mathcal{I}$ choose $\gamma_i \in F(X_i)$, and consider the partial ordering of $\mathcal{I}$ given by $i \geq j$ precisely if there exists a morphism $f_{i,j} : X_i \to X_j$. Given such a morphism $f_{i,j} : X_i \to X_j$ we may assume without loss of generality that $f_{i,j}(\gamma_i) = \gamma_j$ by postcomposition with an automorphism of X_j, in which case the morphism $f_{i,j} : X_i \to X_j$ is uniquely determined. By construction I is a directed poset since for $i, j \in I$ there exists a Galois object $Y \in \mathcal{C}$ and a morphism $Y \to X_i \times X_j$ in which case $Y \simeq X_k$ for some $k \geq i, j$.

We claim that the functor F is isomorphic to the functor $\widetilde{F}$ given by

$$\widetilde{F}(X) = \varinjlim_{i \in I} \mathbf{Hom}_{\mathcal{C}}(X_i, X)$$

To that end consider the natural transformation $\varphi : \widetilde{F} \to F$ given for each morphism $f : X_i \to X$ by $\varphi_X(f) = F(f)(\gamma_i)$. Now since X and each X_i are connected, for morphisms $f : X_i \to X$ and $g : X_j \to X$ such that $F(f)(x) = F(g)(x)$ for some $x \in F(X)$ we have $f = g$, it follows that φ is injective. To show that φ is surjective let $\gamma \in F(X)$; we may assume that X is Galois in which case there exists some isomorphism $X_i \xrightarrow{\sim} X$ sending γ_i to γ, and so it follows that φ is surjective and hence an isomorphism.

Now let $A_i = \mathbf{Aut}(x_i)$. We claim that for $i \geq j$ there is a canonical morphism $h_{i,j} : A_i \to A_j$ such that for all $\theta \in A_i$ we have the commutative diagram

$$\begin{array}{ccc} X_i & \xrightarrow{f_{i,j}} & X_j \\ \theta\downarrow & & \downarrow h_{i,j}(\theta) \\ X_i & \xrightarrow{f_{i,j}} & X_j \end{array}$$

To that end let $h_{i,j}(\theta) : X_j \to X_j$ be the unique automorphism such that $F(h_{i,j}(\theta))(\gamma_j) = F(f_{i,j} \circ \theta)(\gamma_i)$. Then clearly this unique and yields the above commutative diagram. Then for $i \geq j \geq k$ we have $h_{j,k} \circ h_{i,j} = h_{i,k}$, and since $F(X_i) \to F(X_j)$ is surjective it follows that the induced map $\mathbf{Aut}(X_i) \to \mathbf{Aut}(X_j)$ is surjective. Now consider the limit $A = \varprojlim_{i \in I} A_i$. Then this is a profinite group since each $\mathbf{Aut}(X_i)$ is finite, and each of the morphisms $A \to A_i$ are surjective. Then A acts on the inverse system $\{X_i\}_{i \in I}$ and we obtain an action $A \to \mathbf{Aut}(\widetilde{F})$ by precomposition. But since

G acts on $\{X_i\}_{i \in I}$ by postcomposition we obtain a morphism $A^{\mathbf{op}} \to G$. Since each $A \to A_i$ is surjective it follows that G acts transitively on each $F(X_i)$, and since each connected object X is dominated by one of the X_i the result follows. □

Not only does the universal Galois group $G = \mathbf{Aut}(F)$ of $\mathcal{C}$ act transitively on all connected objects, it completely determines the structure of $\mathcal{C}$ in the sense that we can recover $\mathcal{C}$ from the category $G\mathbf{Fin}$, which is the universal property of Galois categories:

Theorem 2.6. *Let $\mathcal{C}$ be a Galois category with fiber functor F, and let $G = \mathbf{Aut}(F)$ be the profinite automorphism group of F. Then there is an equivalence of categories $\mathcal{C} \simeq G\mathbf{Fin}$.*

Proof. We already know that F is faithful. To show that F is full let $X \in \mathcal{C}$ and consider the decomposition $X \simeq X_1 \amalg ... \amalg X_n$ with each X_i connected, and note that since G acts on each $F(X_i)$ transitively it follows that F preserves the decomposition of X into connected components. Then let $Y \in \mathcal{C}$ and let $\varphi : F(X) \to F(Y)$ be a map. Then its graph $\Gamma_\varphi \subseteq X \times Y$ has a decomposition $\Gamma_\varphi \simeq Z_1 \amalg ... \amalg Z_l$, in which case we have a monomorphism $Z \hookrightarrow X \times Y$ in $\mathcal{C}$ with $F(Z) = \Gamma_\varphi$. But since the map $F(Z) \to F(X)$ is bijective and F reflects isomorphism we have an isomorphism $Z \xrightarrow{\sim} X$ and for $f : X \xrightarrow{\sim} Z \to Y$ the composition we have $\varphi = F(f)$ so it follows that F is fully faithful.

Now it suffices to show that F is essentially surjective. Let H be an open subgroup of finite index in G; by definition of the profinite topology there exists objects $X_1, ..., X_n \in \mathcal{C}$ such that the kernel of the canonical morphism $G \to \prod_{1 \leq i \leq n} \mathbf{Aut}(F(X_i))$ is contained in H. We may assume without loss of generality that each X_i is connected, in which case we can choose a Galois object $Y \in \mathcal{C}$ mapping to a connected component of $\prod_{1 \leq i \leq n} X_i$. Now let U be an open subgroup of finite index in G yielding an isomorphism $F(Y) \simeq G/U$. Since Y is Galois the group $\mathbf{Aut}(Y) = \mathbf{Aut}(G/U)$ acts transitively in $F(Y) \simeq G/U$, in which case U is normal in G. But since $F(Y)$ surjects onto each $F(X_i)$ it follows that $U \subseteq H$. Now let $K \subseteq \mathbf{Aut}(Y)$ be the finite subgroup corresponding to the inclusion $(H/U)^{\mathbf{op}} \subseteq (G/U)^{\mathbf{op}} = \mathbf{Aut}(G/U) = \mathbf{Aut}(Y)$. Then for $X = Y/K$ we have $F(X) = G/H$ since F is exact so it follows that G/H is in the essential image of F for every such H. The result follows since every $X \in G\mathbf{Fin}$ can be written as a finite coproduct $X \simeq G/H_1 \amalg ... \amalg G/H_n$ for each H_i an open subgroup of finite index in G. □

In other words, Galois categories are characterized by the following property, which is what we set out to prove in the first place:

Corollary 2.7. *Let $\mathcal{C}$ be a Galois category with fiber functor F. Then there exists a unique profinite group $\widehat{\pi}_1(\mathcal{C}, F)$ such that there is an equivalence of categories $\mathcal{C} \simeq \widehat{\pi}_1(\mathcal{C}, F)\mathbf{Fin}$.*

We write $\widehat{\pi}_1(\mathcal{C}, F)$ to remind ourselves that this group is profinite to distinguish it from the discrete fundamental group given in the locally simply connected case. In fact, in the situation where $\mathcal{C}$ is the Galois category of finite covers of a locally simply connected space X, then the profinite group $\widehat{\pi}_1(X, x)$ is related to the discrete fundamental group $\pi_1(X, x)$ by profinite completion.

We will end the section with the following important theorem, identifying the profinite completion of the fundamental group of a locally simply connected space with the profinite group obtained from its Galois category of finite unramified covers. Specifically, we immediately obtain the following theorem from the above considerations:

Theorem 2.8. *Let X be a locally simply connected space, let $x \in X$ be a basepoint, and let $F_x : \mathbf{FEt}_X \to \mathbf{Fin}$ be the associated fiber functor. Then for $\widehat{\pi}_1(X, x) = \widehat{\pi}_1(\mathbf{FEt}_X, F_x) = \mathbf{Aut}(F_x)$, we have* *a canonical isomorphism*

$$\widehat{\pi}_1(X,x) \simeq \pi_1(X,x)^\wedge$$

Proof. By definition the profinite completion $\pi_1(X,x)^\wedge$ is the limit $\varprojlim_N \pi_1(X,x)/N$ where N ranges over normal subgroups of finite index in $\pi_1(X,x)$. But the finite quotients $\pi_1(X,x)/N$ each correspond to a finite Galois cover Y of X, in which case we have an isomorphism $\mathbf{Aut}(F_x(Y)) \simeq \pi_1(X,x)/N$. But since every finite cover of X is dominated by a finite Galois cover, it follows that

$$\widehat{\pi}_1(X,x) = \mathbf{Aut}(F_x) \simeq \varprojlim_Y \mathbf{Aut}(F_x(Y)) \simeq \varprojlim_N \pi_1(X,x)/N = \pi_1(X,x)^\wedge \qquad \square$$

This will become relevant for comparison theorems between sites. Specifically, for schemes of finite type over $\mathbb{C}$ we will obtain a canonical isomorphism between the étale fundamental group of X and the profinite completion of the topological fundamental group of X as a consequence of this theorem.

Although the machinery of Galois categories takes some effort to develop, it is not without reward: once we show that the category $\mathbf{FEt}_X$ of schemes finite étale over a base scheme X form a Galois category with respect to the fiber functor $F_{\overline{x}} : \mathbf{FEt}_X \to \mathbf{Fin}$ induced by a geometric point $\overline{x}$, we will have succeeded in defining the étale fundamental group for schemes in general. This is what we do next.

2.2 Étale Morphisms

For many purposes, especially Galois theory, the Zariski topology on schemes is too coarse. On the other hand the étale topology is well-suited for these purposes: étale morphisms are those morphisms which are flat, that is with no jumps, and unramified, that is with no ramification points; in fact this is a reasonable definition, but we will instead take the approach of defining étale morphisms in terms of relative Kähler differentials, from which we can show that étale morphisms are equivalently those which are flat and unramified, as well as other characterizations. For this we consult [Stacks 28.32] and [Stacks 28.34], [Milne], and [Szamuely].

The usual definition begins as follows: let $f : X \to S$ be a separated morphism of schemes, let $\Delta : X \to X \times_S X$ be the diagonal morphism, and let $\mathcal{I}$ be the kernel of the induced morphism $\Delta^\sharp : \mathcal{O}_{X\times_Y X} \to \Delta_* \mathcal{O}_X$. The $\mathcal{O}_X$-module of relative Kähler differentials $\Omega_{X/S}$ is the pullback $\Delta^*(\mathcal{I}/\mathcal{I}^2)$ of $\mathcal{I}/\mathcal{I}^2$ along the canonical isomorphism $\Delta : X \xrightarrow{\sim} \Delta(X)$. In particular for $f : \mathbf{Spec}(A) \to \mathbf{Spec}(R)$ a morphism of affine schemes, $f^\sharp : R \to A$ the induced morphism, and I the kernel of the induced multiplication map $\mu : A \otimes_R A \to A$ given by $\mu(a \otimes b) = ab$, the A-module $\Omega_{A/R}$ of relative Kähler differentials is canonically isomorphic to the quotient I/I^2 regarded as an A-module with respect to the isomorphism $(A \otimes_R A)/I \simeq A$.

Example 2.1. *For $f : \mathbf{Spec}(A) \to \mathbf{Spec}(R)$ a morphism of affine schemes and $f^\sharp : R \to A$ the induced morphism, the A-module $\Omega_{A/R}$ of relative Kähler differentials has the following equivalent description: $\Omega_{A/R}$ is the quotient of the free A-module on elements da for $a \in A$ by the sub-A-module generated by elements dr, $d(a+b) - da - db$, and $d(ab) - (da)b - a(db)$ for $r \in R$ and $a, b \in A$. In particular for $A = R[x_1, ..., x_n]/(f_1, ..., f_m)$, then $\Omega_{A/R}$ is the quotient of the free A-module on elements dx_i for $1 \le i \le n$ by the sub-A-module generated by the elements $\sum_{1\le i\le n} \frac{\partial f_j}{\partial x_i} dx_i$ for $1 \le j \le m$.*

Using relative Kähler differentials, we have a convenient definition for smooth morphisms which quickly leads to the definition of étale morphisms.

Definition 2.3. *Let $f : X \to S$ be a morphism of schemes. Then f is called smooth if for all $x \in X$ and $s = f(x)$ the induced morphism of local rings $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ can be generated by at most $\mathbf{dim}_x(X_{f(x)})$ elements, or equivalently if the $\mathbf{k}(x)$-vector space $\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \mathbf{k}(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \mathbf{k}(x)$ can be generated by at most $\mathbf{dim}_x(X_{f(x)})$ elements.*

This is manifestly a local condition on f. In this situation we say that $f : X \to S$ is smooth of relative dimension d if f is smooth and $\Omega_{X/S}$ is a finite locally free $\mathcal{O}_X$-module of constant rank d. In particular if $f : X \to S$ is smooth of relative dimension 0, we obtain the definition of an étale morphism of schemes:

Definition 2.4. *Let $f : X \to S$ be a morphism of schemes. Then f is said to be étale if f is smooth and if $\Omega_{X/S} = 0$.*

Again this is manifestly a local condition on f. Indeed for $f : X \to S$ locally of finite type, f is étale precisely if for all $x \in X$ and $s = f(x)$ the induced morphism of local rings $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $\Omega_{X/S,x} = 0$, or equivalently if $\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \mathbf{k}(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \mathbf{k}(x) = 0$. Equivalently f is étale if for all $x \in X$ and $s = f(x)$ the induced morphism of local rings $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat, $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$, and the extension of residue fields $\mathbf{k}(x)/\mathbf{k}(s)$ is finite and separable [Stacks 28.34]. In fact this last condition, namely on the ramification of extensions of residue fields, invites a more general definition:

Definition 2.5. *Let $f : X \to S$ be a morphism of schemes. Then f is said to be unramified if f is locally of finite type and $\Omega_{X/S} = 0$.*

In particular if $f : X \to S$ is unramified then for each $x \in X$ and $s = f(x)$ the extension of residue fields $\mathbf{k}(x)/\mathbf{k}(s)$ is finite and separable. Conversely for $f : X \to S$ locally of finite type, f is unramified precisely if for all $s \in S$ the fiber X_s is a disjoint union $X_s \simeq \mathbf{Spec}(L_1) \amalg \dots \amalg \mathbf{Spec}(L_n)$ with each $L_i/\mathbf{k}(s)$ a finite separable extension. In other words $f : X \to S$ is of relative dimension 0 and we obtain the following lemma:

Lemma 2.9. *Let $f : X \to S$ be a morphism of schemes. Then f is étale precisely if it is smooth and unramified.*

Proof. On one hand if f is étale then f is smooth of relative dimension 0 hence locally of finite type, and $\Omega_{X/S} = 0$ hence f is unramified. On the other hand if f is smooth and unramified then f is smooth and $\Omega_{X/S} = 0$ and hence f is étale. □

Corollary 2.10. *Let $f : X \to S$ be an étale morphism of schemes. Then f is flat and open, hence flat and unramified. Conversely, let $f : X \to S$ be a flat and unramified morphism of schemes; then f is étale.*

Perhaps the most crucial example of étale morphisms are those schemes finite étale over $\mathbf{Spec}(k)$ for k a field. In this situation the schemes finite étale over $\mathbf{Spec}(k)$ are of the form $\mathbf{Spec}(L)$ for L/k a finite separable extension:

Example 2.2. *Let k be a field. On one hand for L/k a finite separable extension the induced morphism $\mathbf{Spec}(L) \to \mathbf{Spec}(k)$ is finite étale since this morphism is flat and since for the unique closed point $s \in \mathbf{Spec}(k)$ the fiber X_s is isomorphic to $\mathbf{Spec}(L)$ it follows that $\mathbf{Spec}(L) \to \mathbf{Spec}(k)$ is finite étale. On the other hand for X a scheme finite and étale over $\mathbf{Spec}(k)$, then for any affine open subscheme U of X we have $U = \mathbf{Spec}(k_1) \amalg \dots \amalg \mathbf{Spec}(k_n)$ with each k_i/k finite and separable, but every point of*

X is closed since X is finite étale over X, so $X \simeq \mathbf{Spec}(K_1) \amalg ... \amalg \mathbf{Spec}(K_N)$ with each K_i/k finite and separable.

In particular for $\overline{k}$ a separably closed field we have an equivalence of categories $\mathbf{FEt}_{\overline{k}} \simeq \mathbf{Fin}$ since every finite separable extension of $\overline{k}$ is the trivial extension and hence every scheme finite étale over $\mathbf{Spec}(\overline{k})$ is isomorphic to a finite coproduct of $\mathbf{Spec}(\overline{k})$. This will be particularly important in the construction of the fiber functor making $\mathbf{FEt}_X$ into a Galois category,

Another useful way of determining whether a morphism $f : X \to Y$ of (projective, affine) varieties over a field k is étale is the (projective, affine) tangent cone of f.

Example 2.3. *Let X be an affine variety over a field k defined by an ideal $I \subseteq k[x_1, ..., x_n]$. Let $I_0 = \{f_0 \mid f \in I\}$ where f_0 is the homogeneous component of lowest degree in f. Then the tangent cone at the origin is the affine scheme $C_0(X) = \mathbf{Spec}(k[x_1, ..., x_n]/I_0)$. More generally for $x \in X$ the tangent cone $C_x(X)$ is obtained by appropriate coordinate transformation. Then a morphism $f : X \to Y$ of affine varieties over an algebraically closed field $\overline{k}$ is étale at a point $x \in X$ precisely if it induces an isomorphism $C_x(X) \xrightarrow{\sim} X_{fx}(Y)$ of tangent cones.*

For instance let X be the cuspidal cubic given by $y^2 - x^3 = 0$ in $\mathbb{A}^2_k$, and let $f : X \to \mathbb{A}^1_k$ be the projection onto the x-coordinate. Then for $x \in X$ the point above $0 \in \mathbb{A}^1_k$ we have $C_0(\mathbb{A}^1_k) \simeq \mathbb{A}^1_k$ whereas $C_x(X) \simeq \mathbf{Spec}(k[x, y]/(y^2))$ has a nonreduced point so f is not étale above 0. Similarly let X be the nodal cubic given by $y^2 - x^3 + x^2 = 0$ in $\mathbb{A}^2_k$, and let $f : X \to \mathbb{A}^1_k$ be the projection onto the x-coordinate. Then for $x \in X$ the point above $0 \in \mathbb{A}^1_k$ we have $C_0(\mathbb{A}^1_k) \simeq \mathbb{A}^1_k$ whereas $C_x(X) \simeq \mathbf{Spec}(k[x, y]/(xy))$ is reducible so f is not étale above 0.

Indeed the category of schemes finite and étale over a base scheme X is morally the correct category of finite covering spaces of X: such schemes are flat and unramified over X and hence have no jump or branch points, and such schemes are finite over X and hence are finite as covers. We claim that $\mathbf{FEt}_X$ is a Galois category with respect to the obvious fiber functor $F_{\overline{x}} : \mathbf{FEt}_X \to \mathbf{Fin}$ induced by a geometric point $\overline{x}$, and hence defines a profinite group $\widehat{\pi}_1(X, \overline{x})$ yielding an equivalence of categories $\mathbf{FEt}_X \simeq \widehat{\pi}_1(X, \overline{x})\mathbf{Fin}$, and in particular yielding a bijection between conjugacy classes of open subgroups of $\widehat{\pi}_1(X, \overline{x})$ and isomorphism classes of connected schemes finite étale over X. Indeed it suffices to verify the defining properties of a Galois category, which were deliberately chosen in order to make this verification easy to check.

We must first show that $\mathbf{FEt}_X$ has finite limits and that the fiber functor $F : \mathbf{FEt}_X \to \mathbf{Fin}$ preserves these. In fact we will show somewhat more, that for $f : X \to Y$ a morphism of schemes the base change $f_* : \mathbf{FEt}_X \to \mathbf{FEt}_Y$ is left exact.

Lemma 2.11. *Let X be a scheme. Then $\mathbf{FEt}_X$ has finite limits, and for $f : X \to Y$ a morphism of schemes the base change $f_* : \mathbf{FEt}_X \to \mathbf{FEt}_Y$ is left exact.*

Proof. It suffices to show that $\mathbf{FEt}_X$ has a terminal object and pullbacks, and that for $f : X \to Y$ a morphism of schemes the base change $f_* : \mathbf{FEt}_X \to \mathbf{FEt}_Y$ preserves these. But this is obvious: the identity $1_X : X \to X$ is terminal in $\mathbf{FEt}_X$, and the pullback of a finite étale morphism along a finite étale morphism is again a finite étale morphism. Moreover both of these commute with base change by functoriality and by pasting of pullbacks, and hence the base change $f_* : \mathbf{FEt}_X \to \mathbf{FEt}_Y$ is left exact. □

Dually we must show that $\mathbf{FEt}_X$ has finite colimits and that the fiber functor $F : \mathbf{FEt}_X \to \mathbf{Fin}$ preserves these. Again we will show somewhat more, that for $f : X \to Y$ a morphism of schemes the induced functor $f_* : \mathbf{FEt}_X \to \mathbf{FEt}_Y$ is right exact, that is it preserves finite colimits. The case of colimits is somewhat more difficult, as we need a result about étale descent in order to construct the appropriate coequalizers.

Specifically, we will use the fact that for $f : Y \to X$ a morphism of schemes, the properties of f being finite and separated are étale local conditions on the base scheme X. In this case for $Y \to X$ a finite étale covering we can replace X by members of an étale covering so that $Y = \coprod_{1 \le i \le n} X$ is the trivial cover. We will use this in constructing coequalizers in $\mathbf{FEt}_X$, which we can construct dually as an equalizer of finite locally free $\mathcal{O}_X$-algebras, and then applying étale descent until the result is finite étale over X.

Lemma 2.12. *Let X be a scheme. Then $\mathbf{FEt}_X$ has finite colimits, and for $f : X \to Y$ a morphism of schemes the base change $f_* : \mathbf{FEt}_X \to \mathbf{FEt}_Y$ is right exact.*

Proof. It suffices to show that $\mathbf{FEt}_X$ has finite coproducts and coequalizers, and that for $f : X \to Y$ a morphism of schemes the base change $f_* : \mathbf{FEt}_X \to \mathbf{FEt}_Y$ preserves these. The former is obvious since coproducts in $\mathbf{FEt}_X$ are given by disjoint union which is clearly finite étale, and moreover the base change preserves these by the universal property. Now let $f, g : U \to V$ be morphisms in $\mathbf{FEt}_X$; since U and V are finite étale over X there exist finite locally free $\mathcal{O}_X$-algebras $\mathcal{A}$ and $\mathcal{B}$ such that $U = \mathbf{Spec}(\mathcal{A})$ and $V = \mathbf{Spec}(\mathcal{B})$. Then $f, g : U \to V$ induce morphisms $f^\sharp, g^\sharp : \mathcal{B} \to \mathcal{A}$ of $\mathcal{O}_X$-algebras; let $\mathcal{C} = \mathbf{eq}(f^\sharp, g^\sharp)$ be their equalizer. Then if $\mathbf{Spec}(\mathcal{C}) \to X$ is finite étale we are done, otherwise by étale descent we can replace X by a member of an étale cover, in which case we may assume that $U = \coprod_{1 \le i \le n} X$ and $V = \coprod_{1 \le i \le m} X$ so that $\mathcal{A} = \prod_{1 \le i \le n} \mathcal{O}_X$ and $\mathcal{B} = \prod_{1 \le i \le m} \mathcal{O}_X$. After further replacement by members of an open cover if necessary we may assume that f and g correspond to the morphisms $f^\sharp, g^\sharp : \prod_{1 \le i \le n} \mathcal{O}_X \to \prod_{1 \le i \le m} \mathcal{O}_X$ induced by permutation of factors $\sigma : \{1, ..., n\} \rightrightarrows \{1, ..., m\}$. Now consider the coequalizer

$$\prod_{1 \le i \le n} \mathcal{O}_X \underset{g^\sharp}{\overset{f^\sharp}{\rightrightarrows}} \prod_{1 \le i \le m} \mathcal{O}_X \longrightarrow \prod_{i \in I} \mathcal{O}_X$$

Then $\mathcal{C} = \prod_{i \in I} \mathcal{O}_X$ is such that $\mathbf{Spec}(\mathcal{C}) \to X$ is finite étale, and is the desired coequalizer. Moreover for $f : X \to Y$ a morphism of schemes the base change $f_* : \mathbf{FEt}_X \to \mathbf{FEt}_Y$ preserves this, since the above equalizer in the category of $\mathcal{O}_X$-algebras is natural under pushforward, so the required coequalizer is natural under pullback and hence is preserved by base change and hence $f_* : \mathbf{FEt}_X \to \mathbf{FEt}_Y$ is right exact. □

Having shown that $\mathbf{FEt}_X$ has finite limits and finite colimits, and that the base change for any morphism of schemes is exact, we get for free that the fiber functor $F_{\overline{x}} : \mathbf{FEt}_X \to \mathbf{Fin}$ is exact, in which case it suffices to show the remaining two properties.

Theorem 2.13. *Let X be a connected scheme with a geometric point $\overline{x} : \mathbf{Spec}(\overline{k}) \to X$. Then the induced fiber functor $F_{\overline{x}} : \mathbf{FEt}_X \to \mathbf{Set}$ defines a Galois category.*

Proof. Since $\overline{k}$ is algebraically closed we have an equivalence of categories $\mathbf{FEt}_{\mathbf{Spec}(\overline{k})} \simeq \mathbf{Fin}$. Then the fiber functor $F_{\overline{x}}$ is equivalently the base change of schemes finite étale over X along $\overline{x}$ which is exact.

To show that every finite étale scheme is a finite coproduct of its connected components, let $f : Z \to Y$ be a monomorphism in $\mathbf{FEt}_X$; then the canonical morphism $Z \to Z \times_Y Z$ is an isomorphism and hence $Z \to Y$ is a monomorphism of schemes hence an open immersion of schemes. Since Z is finite over X and Y is separated over X it follows that $f : Z \to Y$ is finite hence a closed immersion. But then Y is a connected object of $\mathbf{FEt}_X$ precisely if it is connected as a scheme, hence Y can be written as a coproduct $Y \simeq Y_1 \amalg \ldots \amalg Y_n$ with each Y_i connected in $\mathbf{FEt}_X$.

Finally to show that $F_{\overline{x}}$ reflects isomorphisms let $f : Z \to Y$ be a morphism in $\mathbf{FEt}_X$ such that $F_{\overline{x}}(f)$ is an isomorphism. We may assume that Y is connected, in which case it suffices to show that f is finite locally free of degree 1 and hence an isomorphism. But this is certainly true for any étale neighborhood of Y over $\overline{x}$ and since Y is connected and degree is locally constant the result follows. □

2.3 The Étale Fundamental Group

Since we have shown that the category of schemes finite étale over a base scheme is a Galois category with respect to the fiber functor induced by a geometric point $\overline{x} : \overline{k} \to X$, we can use these data to define a profinite group $\widehat{\pi}_1(X, \overline{x})$ which identifies schemes finite and étale over X with its finite permutation representations. Since this formulation follows immediately from our consideration of Galois categories following [Stacks 52.5] we will largely focus on examples of étale fundamental groups in geometry and number theory; for this we consult [Oort] and [Matsumoto].

Definition 2.6. *Let X be a connected scheme with a geometric point $\overline{x}$ and let $F_{\overline{x}} : \mathbf{FEt}_X \to \mathbf{Fin}$ be the induced fiber functor. Then the étale fundamental group $\widehat{\pi}_1(X, \overline{x})$ of X at $\overline{x}$ is the automorphism group $\mathbf{Aut}(F_{\overline{x}})$.*

Since $\mathbf{FEt}_X$ is a Galois category it follows that $\widehat{\pi}_1(X, \overline{x}) = \mathbf{Aut}(F_{\overline{x}})$ is profinite and that we have an equivalence of categories $\mathbf{FEt}_X \simeq \widehat{\pi}_1(X, \overline{x})\mathbf{Fin}$.

There are two additional ways this construction is functorial. On one hand $f : X \to Y$ a morphism of schemes mapping a geometric point $\overline{x} : \mathbf{Spec}(\Omega) \to X$ to a geometric point $\overline{y} : \mathbf{Spec}(\Omega) \to Y$ the induced morphism $f_* : \widehat{\pi}_1(X, \overline{x}) \to \widehat{\pi}_1(Y, \overline{y})$ yields a functor $f^* : \widehat{\pi}_1(Y, \overline{y})\mathbf{Fin} \to \widehat{\pi}_1(X, \overline{x})\mathbf{Fin}$ which commutes with the (exact) pullback functor $f^* : \mathbf{FEt}_Y \to \mathbf{FEt}_X$ yielding a commutative diagram

$$\begin{array}{ccc} \mathbf{FEt}_Y & \xrightarrow{\mathbf{Fib}_{\overline{y}}} & \widehat{\pi}_1(Y, \overline{y})\mathbf{Fin} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbf{FEt}_X & \xrightarrow{\mathbf{Fib}_{\overline{x}}} & \widehat{\pi}_1(X, \overline{x})\mathbf{Fin} \end{array}$$

On the other hand for $\overline{x}, \overline{y} : \mathbf{Spec}(\Omega) \to X$ two geometric points of a connected scheme X we obtain a natural isomorphism of fiber functors $F_{\overline{x}} \simeq F_{\overline{y}} : \mathbf{FEt}_X \to \mathbf{Fin}$ which yields an isomorphism $\widehat{\pi}_1(X, \overline{x}) \simeq \widehat{\pi}_1(X, \overline{y})$, although this isomorphism is not canonical.

The first crucial example of an étale fundamental group is that of $\mathbf{Spec}(k)$ for k a field. In this case the étale fundamental group of $\mathbf{Spec}(k)$ will be isomorphic to the absolute Galois group of k. We have seen that finite étale covers of $\mathbf{Spec}(k)$ are of the form $\mathbf{Spec}(k_1) \amalg \ldots \amalg \mathbf{Spec}(k_n)$ with each k_i/k finite and separable; conversely the finite separable extensions of a fields are precisely those connected schemes finite étale over $\mathbf{Spec}(k)$ which are identified with normal subgroups of the absolute Galois group:

Lemma 2.14. *Let k be a field with a fixed separable closure $k^{\mathbf{s}}$. Then a separable extension L/k is Galois precisely if $\mathbf{Aut}(k^{\mathbf{s}}/L)$ is normal in $\mathbf{Gal}(k^{\mathbf{s}}/k)$, in which case there is a canonical isomorphism $\mathbf{Gal}(L/k) \simeq \mathbf{Gal}(k^{\mathbf{s}}/k)/\mathbf{Aut}(k^{\mathbf{s}}/L)$.*

Proof. It suffices to check that L/k is splitting; to that end L/k is splitting precisely if for all $\sigma \in \mathbf{Gal}(k^{\mathbf{s}}/k)$ and $x \in L$ we have $\sigma(x) \in L$, precisely if for all $\sigma \in \mathbf{Gal}(k^{\mathbf{s}}/k)$, $\tau \in \mathbf{Aut}(k^{\mathbf{s}}/L)$, $x \in L$ we have $\tau(\sigma(x)) = \sigma(x)$, or equivalently $(\sigma\tau\sigma^{-1})(x) = x$ so $\sigma\tau\sigma^{-1} \in \mathbf{Aut}(k^{\mathbf{s}}/L)$ hence $\mathbf{Aut}(k^{\mathbf{s}}/L)$ is normal. Now consider the restriction map

$$\mathbf{res}_L : \mathbf{Gal}(k^{\mathbf{s}}/k) \to \mathbf{Gal}(L/k)$$
$$\sigma \mapsto \sigma|_L$$

Then $\mathbf{res}_L$ is surjective with kernel $\mathbf{Aut}(k^{\mathbf{s}}/L)$ so we have a short exact sequence

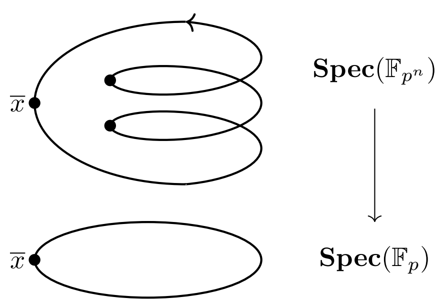
$$0 \to \mathbf{Gal}(k^{\mathbf{s}}/L) \to \mathbf{Gal}(k^{\mathbf{s}}/k) \to \mathbf{Gal}(L/k) \to 0$$

isomorphism $\mathbf{Gal}(L/k) \simeq \mathbf{Gal}(k^{\mathbf{s}}/k)/\mathbf{Aut}(k^{s}/L)$. □

Corollary 2.15. *Let k be a field with fixed separable closure $k^{\mathbf{s}}$ and fixed algebraic closure $\overline{k}$ corresponding to the geometric point $\overline{k} : \mathbf{Spec}(\overline{k}) \to \mathbf{Spec}(k)$. Then we have an isomorphism of profinite groups $\widehat{\pi}_1(\mathbf{Spec}(k), \overline{k}) \simeq \mathbf{Gal}(k^{\mathbf{s}}/k)$.*

Already this creates a wealth of examples, not least the absolute Galois group $\widehat{\pi}_1(\mathbf{Spec}(\mathbb{Q}), \overline{x}) \simeq \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and its maximal Abelian quotient $\widehat{\pi}_1^{\mathbf{ab}}(\mathbf{Spec}(\mathbb{Q}), \overline{x}) \simeq \mathbf{Gal}(\mathbb{Q}^{\mathbf{ab}}/\mathbb{Q}) \simeq \widehat{\mathbb{Z}}^{\times}$ described by class field theory, and more generally the absolute Galois groups and maximal Abelian quotients of number fields. One important example is the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}$ of a finite field generated by the Frobenius; although $\mathbf{Spec}(\mathbb{F}_p)$ is geometrically a point, its finite covering theory behaves similarly to that of the circle:

Example 2.4. *Fix a prime p and let $k = \mathbb{F}_p$. For each $n \in \mathbb{N}$ consider the extension $\mathbb{F}_{p^n}$ of $\mathbb{F}_p$ given as the splitting field of $x^{p^n} - x \in \mathbb{F}_p[x]$ which has degree n with cyclic Galois group $\mathbb{Z}/n\mathbb{Z}$ generated by the Frobenius $\mathbf{Frob}(x) = x^p$. Indeed every finite separable extension of $\mathbb{F}_p$ is of this form, and the algebraic closure $\overline{\mathbb{F}}_p$ is obtained as the inverse limit $\overline{\mathbb{F}}_p \simeq \varprojlim_{n\in\mathbb{N}} \mathbb{F}_{p^n}$ with absolute Galois group $\mathbf{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}$ the free profinite group on a single generator. In this way we can think of $\mathbf{Spec}(\mathbb{F}_p)$ as a profinite S^1 with monodromy given by the Frobenius: although each scheme $\mathbf{Spec}(\mathbb{F}_{p^n})$ has only a single point, the fiber $F_{\overline{x}}(\mathbb{F}_{p^n}) = \mathbf{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^n}, \overline{\mathbb{F}}_p) = \{\overline{x}, \mathbf{Frob}(\overline{x}), ..., \mathbf{Frob}^{n-1}(\overline{x})\} \simeq \mathbb{Z}/n\mathbb{Z}$ has the correct number of Frobenius shifted geometric points.*



Indeed $\mathbf{Spec}(\overline{\mathbb{F}}_p)$ exists as a universal cover of $\mathbf{Spec}(\mathbb{F}_p)$ in the category of schemes since $\mathbf{Spec}(\overline{\mathbb{F}}_p)$ is affine and étale simply connected, and every other étale simply connected cover of $\mathbf{Spec}(\mathbb{F}_p)$ factors through $\mathbf{Spec}(\overline{\mathbb{F}}_p)$. Moreover the fiber $F_{\overline{x}}(\overline{F}_p) \simeq \mathbf{Hom}_{\mathbb{F}_p}(\overline{\mathbb{F}}_p, \overline{\mathbb{F}}_p) = \{..., \mathbf{Frob}^{-1}(\overline{x}), \overline{x}, \mathbf{Frob}(\overline{x}), ...\} \simeq \mathbb{Z}$ is as expected of a universal cover.

The other important example of étale fundamental groups are those of integral normal schemes; in this case the étale fundamental group is identified with the Galois group of a maximal unramified extension of the function field K, since schemes finite étale over such an integral normal scheme X are precisely those schemes $Y \to X$ with function field L for which the normalization of X in L is unramified over X. We borrow a rather general lemma regarding the étale fundamental group of a geometrically unibranch scheme at the generic point, which says that the absolute Galois group of the residue field of the generic point of such a scheme surjects onto its étale fundamental group:

Lemma 2.16. *Let X be an irreducible geometrically unibranch scheme. Then for any nonempty open subscheme U of X and $\overline{u} : \mathbf{Spec}(\Omega) \to U$ a geometric point the induced morphism $\widehat{\pi}_1(U, \overline{u}) \to \widehat{\pi}_1(X, \overline{u})$ is surjective. Moreover for $\eta \to X$ the generic point of X the canonical morphism $\mathbf{Gal}(\mathbf{k}(\eta)^{\mathbf{s}}/\mathbf{k}(\eta)) \simeq \widehat{\pi}_1(\eta, \overline{\eta}) \to \widehat{\pi}_1(X, \overline{\eta})$ is surjective.*

Once we have identified the étale fundamental group of such a geometrically unibranch scheme at the generic point, the claim regarding integral normal schemes is almost immediate: we only need to identify the above surjective map with the surjective morphism $\mathbf{Gal}(K^{\mathbf{s}}/K) \to \mathbf{Gal}(K^{\mathbf{ur}}/K)$ of Galois groups of function fields, from which we obtain the desired isomorphism:

Theorem 2.17. *Let X be an integral normal scheme with function field K and generic point $\eta \to X$. Then the canonical morphism*

$$\mathbf{Gal}(K^{\mathbf{s}}/K) \to \widehat{\pi}_1(\eta, \overline{\eta}) \to \widehat{\pi}_1(X, \overline{\eta})$$

is identified with the quotient map $\mathbf{Gal}(K^{\mathbf{s}}/K) \to \mathbf{Gal}(K^{\mathbf{ur}}/K)$ where $K^{\mathbf{ur}}$ is the maximal algebraic extension of K which is everywhere unramified, hence the union of all finite extensions L of K such that the normalization of X in L is unramified over X.

Proof. Since X is normal it is geometrically unibranch, so by the previous lemma the canonical morphism $\mathbf{Gal}(K^{\mathbf{s}}/K) \simeq \widehat{\pi}_1(\eta, \overline{\eta}) \to \widehat{\pi}_1(X, \overline{\eta})$ is surjective, and moreover the induced functor $\mathbf{FEt}_X \to \mathbf{FEt}_\eta$ is fully faithful. Now consider the functor $\mathbf{FEt}_X \to \mathbf{Gal}(K^{\mathbf{s}}/K)\mathbf{Fin}$ obtained by composing the above functor with the equivalence of categories $\mathbf{FEt}_\eta \xrightarrow{\sim} \mathbf{Gal}(K^{\mathbf{s}}/K)\mathbf{Fin}$. We claim that the essential image of this functor consists of those finite permutation representations of $\mathbf{Gal}(K^{\mathbf{s}}/K)$ of the form $S = \coprod_{i \in I} \mathbf{Hom}_K(L_i, K^{\mathbf{s}})$ with each L_i/K finite separable such that X is unramified in L_i. To see this we simply note that the essential image of this functor is exact the category of such S such that the canonical $\mathbf{Gal}(K^{\mathbf{s}}/K)$-action factors through the surjection $\mathbf{Gal}(K^{\mathbf{s}}/K) \to \widehat{\pi}_1(X, \overline{\eta})$ which has kernel $\mathbf{Gal}(K^{\mathbf{s}}/K^{\mathbf{ur}})$ a normal subgroup of $\mathbf{Gal}(K^{\mathbf{s}}/K)$. But then since $K^{\mathbf{ur}}/K$ is Galois we have a short exact sequence of profinite groups

$$0 \to \mathbf{Gal}(K^{\mathbf{s}}/K^{\mathbf{ur}}) \to \mathbf{Gal}(K^{\mathbf{s}}/K) \to \mathbf{Gal}(K^{\mathbf{ur}}/K) \to 0$$

So it follows that we have an isomorphism $\widehat{\pi}_1(X, \overline{\eta}) \simeq \mathbf{Gal}(K^{\mathbf{ur}}/K)$. □

Corollary 2.18. *Let X be an integral normal scheme with function field K and generic point $\eta \to X$, and let $K^{\mathbf{ur}}$ be the maximal algebraic extension of K which is everywhere unramified. Then we have a canonical isomorphism $\widehat{\pi}_1(X, \overline{\eta}) \simeq \mathbf{Gal}(K^{\mathbf{ur}}/K)$ is an isomorphism.*

Example 2.5. *Consider the scheme $\mathbf{Spec}(\mathbb{Z})$. Since $\mathbf{Spec}(\mathbb{Z})$ is normal every scheme finite étale over $\mathbf{Spec}(\mathbb{Z})$ is of the form $\mathbf{Spec}(\mathcal{O}_K)$ for K a number field, since $\mathcal{O}_K$ is equivalently the normalization of $\mathbb{Z}$ in the number field K over $\mathbb{Q}$. Now suppose that K has degree $n = r_1 + 2r_2$ with r_1 and r_2 the* *number of real and complex embeddings of K respectively, and let Δ_K be the discriminant of K. By the Minkowski bound [Neukerich] every class in the ideal class group $\mathbf{Cl}(K)$ of K contains an integral ideal of norm at most the Monkowski constant M_K of K given by*

$$M_K = \sqrt{|\Delta_K|}\Big(\frac{4}{\pi}\Big)^{r_2}\frac{n!}{n^n}$$

But by definition an integral ideal of $\mathcal{O}_K$ has norm at least 1 so we have $1 \leq M_K$ and hence

$$\sqrt{|\Delta_K|} \geq \Big(\frac{\pi}{4}\Big)^{r_2}\frac{n^n}{n!} \geq \Big(\frac{\pi}{4}\Big)^{n/2}\frac{n^n}{n!}$$

In particular if K has degree $n > 1$ then we have $|\Delta_K| > 1$. Since a prime $p \in \mathbb{Z}$ ramifies in K precisely if p divides Δ_K, it follows that $\mathbf{Spec}(\mathcal{O}_K) \to \mathbf{Spec}(\mathbb{Z})$ is somewhere ramified unless $K = \mathbb{Q}$ and hence $\widehat{\pi}_1(\mathbf{Spec}(\mathbb{Z}), \overline{x}) = 0$.

In general there is less that we can say about $\widehat{\pi}_1(\mathbf{Spec}(\mathcal{O}_K), \overline{x})$ for K a number field. On one hand by class field theory the maximal Abelian quotient $\widehat{\pi}_1^{\mathbf{ab}}(\mathbf{Spec}(\mathcal{O}_K), \overline{x})$ is isomorphic to the narrow class group $\mathbf{Cl}_K^+ = I_K/P_K^+ \simeq \mathbf{Gal}(H_K^+/K)$ where H_K^+ is the narrow class field of K, and I_K/P_K^+ is the group of fractional ideals of $\mathcal{O}_K$ modulo the group of principal fractional ideals $(\alpha) = \alpha\mathcal{O}_K$ whose generator $\alpha \in K$ is totally positive for each real place of $\mathcal{O}_K$. On the other hand anything more than the maximal Abelian quotient of $\widehat{\pi}_1(\mathbf{Spec}(\mathcal{O}_K), \overline{x})$ is poorly understood: by [Golod-Shafarevich] there exists number fields whose class field tower is infinite, and by [Maire] there even exists number fields with class number 1 but nevertheless admit an infinite unramified extension.

Remark 2.1. *To the extent that we can compactify $\mathbf{Spec}(\mathcal{O}_K)$ by adjoining the finite set S_∞ of Archimedean places of K, the maximal Abelian quotient $\widehat{\pi}_1^{\mathbf{ab}}(\mathbf{Spec}(\mathcal{O}_K) \cup S_\infty, \overline{x})$ might be though of as the ideal class group $\mathbf{Cl}_K = I_K/P_K \simeq \mathbf{Gal}(H_K/K)$ where I_K/P_K is the group of fractional ideals of $\mathcal{O}_K$ modulo the group of principal fractional ideals, and H_K is the Hilbert class field of K, that is the maximal Abelian extension of K which is unramified at every place of K. Indeed the ideal class group $\mathbf{Cl}_K$ and the narrow class group $\mathbf{Cl}_K^+$ are related by the short exact sequence*

$$0 \to P_K/P_K^+ \to \mathbf{Cl}_K^+ \to \mathbf{Cl}_K \to 0$$

so that P_K/P_K^+ measures the extent to which some finite Abelian extension of K unramified at every finite place of K fails to be unramified at the Archimedean places of K, or rather the extent to which the appropriate morphism $\widehat{\pi}_1^{\mathbf{ab}}(\mathbf{Spec}(\mathcal{O}_K), \overline{x}) \to \widehat{\pi}_1^{\mathbf{ab}}(\mathbf{Spec}(\mathcal{O}_K) \cup S_\infty, \overline{x})$ induced by the inclusion $\mathbf{Spec}(\mathcal{O}_K) \hookrightarrow \mathbf{Spec}(\mathcal{O}_K) \cup S_\infty$ fails to be an isomorphism. Of course one must appeal to Arakelov theory in order to make sense of the compactification $\mathbf{Spec}(\mathcal{O}_K) \cup S_\infty$ and its étale fundamental group.

Even more generally if we allow for ramification at finite primes in $\mathbf{Spec}(\mathbb{Z})$ the resulting étale fundamental group describes those number fields which are unramified away from these primes.

Example 2.6. *Consider $\mathbf{Spec}(\mathbb{Z}[\frac{1}{n}])$. Then connected schemes finite étale over $\mathbf{Spec}(\mathbb{Z}[\frac{1}{n}])$ are of the form $\mathcal{O}_K[\frac{1}{n}]$ for $\mathcal{O}_K$ the ring of integers of some number field K unramified away from n. Indeed for each such connected scheme X finite étale over $\mathbf{Spec}(\mathbb{Z}[\frac{1}{n}])$ there exists a connected Galois scheme Y finite étale over X, which is of the form $\mathbf{Spec}(\mathcal{O}_L[\frac{1}{n}])$ for $\mathcal{O}_L$ the ring of integers of some Galois number field unramified away from n. But then for $\mathbb{Q}^{(n)}$ the maximal algebraic extension of $\mathbb{Q}$ unramified away from n, it follows that $\widehat{\pi}_1(\mathbf{Spec}(\mathbb{Z}[\frac{1}{n}]), \overline{x}) \simeq \mathbf{Gal}(\mathbb{Q}^{(n)}/\mathbb{Q})$. In particular by class field theory the maximal Abelian quotient $\widehat{\pi}_1^{\mathbf{ab}}(\mathbf{Spec}(\mathbb{Z}[\frac{1}{n}]), \overline{x})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$ as the narrow ray class field of (n) is generated by the n-th roots of unity, see [Milne] and [Neukirch].*

In general for K a number field and $\mathfrak{a}$ an ideal of $\mathcal{O}_K$, we have $\widehat{\pi}_1(\mathbf{Spec}(\mathcal{O}_K[\frac{1}{\mathfrak{a}}]), \overline{x}) \simeq \mathbf{Gal}(K^{(\mathfrak{a})}/K)$, where $K^{(\mathfrak{a})}$ is a maximal algebraic extension of K unramified away from $\mathfrak{a}$. Again there is little we can say about the structure of $\widehat{\pi}_1(\mathbf{Spec}(\mathcal{O}_K[\frac{1}{\mathfrak{a}}]), \overline{x})$. By class field theory the maximal Abelian quotient $\widehat{\pi}_1^{\mathbf{ab}}(\mathbf{Spec}(\mathcal{O}_K[\frac{1}{\mathfrak{a}}]), \overline{x})$ is isomorphic to the narrow ray class group $\mathbf{Cl}^+_{K,\mathfrak{a}} = I^+_{K,\mathfrak{a}}/P^+_{K,\mathfrak{a}} \simeq \mathbf{Gal}(H^+_{K,\mathfrak{a}}/K)$ where $H^+_{K,\mathfrak{a}}$ is the narrow ray class field of K and $I^+_{K,\mathfrak{a}}/P^+_{K,\mathfrak{a}}$ is the group of fractional ideals coprime to $\mathfrak{a}$ moduli the group of principal fractional ideals $(\alpha) = \alpha\mathcal{O}_K$ whose generator $\alpha \in K$ is totally positive for every real place of $\mathcal{O}_K$ and satisfies $\alpha \equiv 1 \mod \mathfrak{a}$. Still anything more than the maximal Abelian quotient of $\widehat{\pi}_1(\mathbf{Spec}(\mathcal{O}_K[\frac{1}{\mathfrak{a}}]), \overline{x})$ is poorly understood.

We can also consider geometric examples of étale fundamental groups, say those of varieties over an algebraically closed field. In this situation the resulting finite covering theory is very similar to what one would find in the topological case, at least in characteristic 0. For instance, the étale fundamental group of the projective and affine spaces over an algebraically closed field of characteristic 0 are trivial since every connected finite étale cover over these varieties is trivial.

Example 2.7. *Let $\overline{k}$ be an algebraically closed field $\overline{k}$ of characteristic 0 and consider the affine line $\mathbb{A}^1_{\overline{k}}$. Recall that for X an integral proper normal curve of genus g_X over $\overline{k}$ and for $Y \to X$ a branched cover of genus g_Y and degree n with ramification indices e_y, we have by the Riemann-Hurewitz formula*

$$2g_Y - 2 = n(2g_X - 2) + \sum_{y\in Y}(e_y - 1)$$

Now let $X \to \mathbb{A}^1_{\overline{k}}$ be a connected finite étale cover of degree n, and let $\overline{X} \to \mathbb{P}^1_{\overline{k}}$ be the corresponding branched cover with ramification of degree at most n over ∞. Then by Riemann-Hurwitz we have $2g_X - 2 = -2n + e_\infty - 1 \leq -n - 1$, a contradiction unless $n = 1$. It follows that every connected finite étale cover of $\mathbb{A}^1_{\overline{k}}$ is trivial and hence $\widehat{\pi}_1(\mathbb{A}^1_{\overline{k}}, \overline{x}) = 0$. Similarly we have $\widehat{\pi}_1(\mathbb{P}^1_{\overline{k}}, \overline{x}) = 0$ since every finite étale cover of $\mathbb{P}^1_{\overline{k}}$ restricts to a finite étale cover of $\mathbb{A}^1_{\overline{k}}$.

More generally for $n > 1$ we have $\widehat{\pi}_1(\mathbb{P}^n_{\overline{k}}, \overline{x}) = 0$ and $\widehat{\pi}_1(\mathbb{A}^n_{\overline{k}}, \overline{x}) = 0$, which is easily seen using the decomposition $\mathbb{P}^n_{\overline{k}} = \mathbb{A}^n_{\overline{k}} \cup \mathbb{P}^{n-1}_{\overline{k}}$ and reducing to the case of $\mathbb{P}^1_{\overline{k}}$ and $\mathbb{A}^1_{\overline{k}}$.

That $\mathbb{A}^n_{\overline{k}}$ and $\mathbb{P}^n_{\overline{k}}$ are étale simply connected matches our intuition: over algebraically closed fields of characteristic 0 we can think of finite étale covers as finite topological covers, not least because of the Lefschetz principle. By contrast, the situation is very different in positive characteristic, so much so that $\mathbb{A}^1_{\overline{k}}$ is not étale simply connected:

Example 2.8. *Let $\overline{k}$ be a field of characteristic $p > 0$. We call any polynomial of the form $f(x) = x^p - x + a \in \overline{k}[x]$ for $a \in \overline{k}$ an Artin-Schreier polynomial; if moreover $a \notin \{y \in \overline{k} \mid y = x^p - x, x \in \overline{k}\}$ then $f(x)$ is irreducible in $\overline{k}[x]$. Then the splitting field K of $f(x)$ is a cyclic Galois extension of order p which is nowhere ramified, and every Galois extension of $\overline{k}$ of degree p equal to the characteristic is in fact the splitting field of some Artin-Schreier polynomial [Lang].*

In particular for $\overline{k}(t)$ a purely transcendental extension of $\overline{k}$, any Artin-Schreier extension $K/\overline{k}(t)$ induces a nontrivial finite étale cover $X \to \mathbb{A}^1_{\overline{k}}$, and hence represents a nontrivial finite permutation representation of $\widehat{\pi}_1(\mathbb{A}^1_{\overline{k}}, \overline{x})$. In particular $\widehat{\pi}_1(\mathbb{A}^1_{\overline{k}}, \overline{x})$ is nontrivial: the characteristic $p > 0$ affine line $\mathbb{A}^1_{\overline{k}}$ is not étale simply connected.

It is worth mentioning that the projective line $\mathbb{P}^1_{\overline{k}}$ over an algebraically closed field of positive characteristic is still étale simply connected: although the existence of Artin-Schreier covers in positive characteristic gives examples of finite étale covers which are unramified outside ∞, every finite étale cover of $\mathbb{P}^1_{\overline{k}}$ must be somewhere ramified as in the characteristic 0 case.

There is another interesting example in characteristic 0 that uses only a mild amount of Kummer theory is enough to calculate the étale fundamental group of the characteristic 0 punctured affine line: one only needs to know that the algebraic closure of $\mathbb{C}((t))$ is the union of its cyclic Galois extensions $\mathbb{C}((\sqrt[n]{t}))/\mathbb{C}((t))$, in which case one can conclude that $\widehat{\pi}_1(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}, \overline{x}) \simeq \widehat{\mathbb{Z}}^{\times}$ which is what one would expect from comparison with the topological fundamental group $\pi_1(\mathbb{C}^{\times}, x) \simeq \mathbb{Z}$.

Example 2.9. *Let $\mathbb{C}((t))$ be the field of Laurent series in a variable t over $\mathbb{C}$. Since for $n \in \mathbb{N}$ the field $\mathbb{C}$ contains the roots of unity ζ_n, we have a finite extension $\mathbb{C}((\sqrt[n]{t}))/\mathbb{C}((t))$ which is cyclic Galois of order n. By Kummer theory we have that $\overline{\mathbb{C}((t))} \simeq \bigcup_{n\in\mathbb{N}} \mathbb{C}((\sqrt[n]{t}))$, and hence it follows that $\widehat{\pi}_1(\mathbf{Spec}(\mathbb{C}((t))), \overline{x}) \simeq \mathbf{Gal}(\overline{\mathbb{C}((t))}/\mathbb{C}((t))) \simeq \widehat{\mathbb{Z}}$, see [Lang].*

Indeed we can use the above to show that $\widehat{\pi}_1(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}, \overline{x}) \simeq \widehat{\mathbb{Z}}$, since each extension $\mathbb{C}((\sqrt[n]{t}))/\mathbb{C}((t))$ induces the map $f : \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ given by $f(t) = t^n$ which we have seen to be étale. Since a finite étale cover of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ is ramified at most over 0, and since every finite étale cover of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ is isomorphic to such a cyclic Galois cover in a neighborhood of 0, it follows that $\widehat{\pi}_1(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}, \overline{x}) \simeq \widehat{\mathbb{Z}}$. In particular it follows for $\overline{k}$ an algebraically closed field of characteristic 0 that $\widehat{\pi}_1(\mathbb{A}^1_{\overline{k}} \setminus \{0\}, \overline{x}) \simeq \widehat{\mathbb{Z}}$.

Still, we need more sophisticated theorems in order to compute étale fundamental groups of general schemes over an algebraically closed field $\overline{k}$ of characteristic 0. For this we appeal to a comparison theorem due to [Grothendieck] and [Grauert-Remmert] which is far outside the scope of this paper: the Grothendieck-Riemann existence theorem shows that the category of finite étale covers of a scheme locally of finite type over $\mathbb{C}$ is equivalent to the category of finite topological covers of the associated complex analytic space, in which case the étale fundamental group of a scheme locally of finite type over $\overline{k}$ is seen to be isomorphic to the profinite completion of the above topological fundamental group by base change to $\mathbb{C}$. More precisely:

Theorem 2.19. *(Grothendieck-Riemann existence theorem) Let X be a scheme locally of finite type over $\mathbb{C}$. Then the assignment of a finite étale cover $Y \to X$ to its analytification $Y^{\mathbf{an}} \to X^{\mathbf{an}}$ induces an equivalence of categories $\mathbf{FEt}(X) \simeq \mathbf{FEt}(X^{\mathbf{an}})$.*

Corollary 2.20. *(Grothendieck) Let X be a scheme locally of finite type over an algebraically closed field $\overline{k}$ of characteristic 0, let $X^{\mathbf{an}}$ be the associated complex analytic space, and let $\overline{x} : \mathbf{Spec}(\Omega) \to X$ be a geometric point. Then the étale fundamental group $\widehat{\pi}_1(X, \overline{x})$ is isomorphic to the profinite completion of the topological fundamental group $\pi_1(X^{\mathbf{an}}, x)$.*

$$\widehat{\pi}_1(X, \overline{x}) \simeq \pi_1(X^{\mathbf{an}}, x)^{\wedge}$$

Proof. This is clear for $\overline{k} = \mathbb{C}$ since the Grothendieck-Riemann existence theorem yields an equivalence of categories $\mathbf{FEt}_{/X} \simeq \mathbf{FEt}_{X^{\mathbf{an}}}$ and hence an isomorphism $\widehat{\pi}_1(X, \overline{x}) \simeq \pi_1(X^{\mathbf{an}}, x)^{\wedge}$. In general the result follows since for Ω an algebraically closed field of characteristic 0 extending $\overline{k}$ the base change induces an equivalence of categories $\mathbf{FEt}_{/X} \simeq \mathbf{FEt}_{/X_{\Omega}}$ and hence an isomorphism $\widehat{\pi}_1(X, \overline{x}) \simeq \pi_1(X^{\mathbf{an}}, x)^{\wedge}$. □

Example 2.10. *Let $\overline{k}$ be an algebraically closed field of characteristic 0, and let $X = \mathbb{P}^1_{\overline{k}} \setminus \{x_0, ..., x_n\}$ or equivalently $X = \mathbb{A}^1_{\overline{k}} \setminus \{x_1, ..., x_n\}$. Then the associated complex analytic space $X^{\mathbf{an}}$ is homotopy equivalent to a wedge sum of n circles, and hence the fundamental group $\pi_1(X^{\mathbf{an}}, x_0) \simeq F_n$ is free on*

n generators represented by simple closed loops $\gamma_1, ..., \gamma_n$ based at x_0 around $x_1, ..., x_n$. It follows that $\widehat{\pi}_1(\mathbb{A}^1_{\overline{k}} \setminus \{x_1, ..., x_n\}, \overline{x}_0) \simeq \widehat{F}_n$ is a free profinite group.

Moreover since $\mathbb{A}^1_{\overline{k}} \setminus \{x_1, ..., x_n\}$ is an integral normal scheme with function field $\overline{k}(t)$, it follows that $\mathbf{Gal}(\overline{k}(t)^{\mathbf{ur}}_S/\overline{k}(t)) \simeq \widehat{F}_n$ where $\overline{k}(t)^{\mathbf{ur}}_S$ is the maximal algebraic extension of $\overline{k}(t)$ which is unramified outside the set S of places corresponding to $x_1, ..., x_n$. In particular it follows that every finite group appears as a Galois group of a finite extension of $\overline{k}(t)$ unramified outside finitely many places.

In fact the above is enough to conclude Douady's theorem on the absolute Galois group of $\overline{k}(t)$. Namely for $\overline{K}$ a fixed algebraic closure of $K = \overline{k}(t)$ we have an isomorphism

$$\widehat{\pi}_1(\mathbf{Spec}(\overline{k}(t)), \overline{x}) \simeq \mathbf{Gal}(\overline{K}/K) \simeq \widehat{F}(\overline{k})$$

where $\widehat{F}(\overline{k})$ is the free profinite group on a set of generators with cardinality equal to that of $\overline{k}$. To see this, we note that finite subsets $S \subseteq T \subseteq \overline{k}$ gives rise to a surjection $\widehat{F}(T) \simeq \widehat{\pi}_1(\mathbb{A}^1_{\overline{k}} \setminus T, \overline{x}) \to \widehat{\pi}_1(\mathbb{A}^1_{\overline{k}} \setminus S, \overline{x}) \simeq \widehat{F}(T)$ which yields an inverse system of free profinite groups with limit $\mathbf{Gal}(\overline{K}/K)$ since every finite subextension of $\overline{K}/K$ is contained in $\overline{k}(t)^{\mathbf{ur}}_S$ for S sufficiently large. It is worth noting that this holds in arbitrary characteristic, see [Haran-Jarden].

Example 2.11. *Let $\overline{X}$ be a integral proper normal curve of genus g over an algebraically closed field of characteristic 0, let $X = \overline{X} \setminus \{x_1, ..., x_n\}$ be an open subscheme, and let $\overline{x} : \mathbf{Spec}(\Omega) \to X$ be a geometric point. It is well known that the associated complex analytic space $X^{\mathbf{an}}$ has fundamental group $\pi_1(X^{\mathbf{an}}, x) \simeq \Pi_{g,n}$ given by the presentation*

$$\Pi_{g,n} = \langle \alpha_1, \beta_1, ..., \alpha_g, \beta_g, \gamma_1, ..., \gamma_n \mid [\alpha_1, \beta_1]...[\alpha_g, \beta_g]\gamma_1...\gamma_n = 1 \rangle$$

Consequently $\widehat{\pi}_1(X, \overline{x})$ is isomorphic to the profinite completion $\widehat{\Pi}_{g,n}$.

On the other hand the structure of étale fundamental groups of schemes over an algebraically closed field of positive characteristic is somewhat more complicated. Still, the resulting étale fundamental groups, say for integral proper normal curves, are exactly the groups one would expect, after completing away from the residue characteristic.

For G a group and p a prime we write $G^{(p')}$ for the prime to p completion given by the limit $\varprojlim_{N \leq G} G/N$ taken over normal subgroups of index prime to p. In particular if G is profinite then its prime to p completion is equivalently the maximal prime to p quotient of G; for instance if $G = \widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell$ then $G^{(p')} = \prod_{\ell \neq p} \mathbb{Z}_\ell$. In this situation Grothendieck's specialization theorem extends the comparison theorem for étale fundamental groups of curves in characteristic 0 to positive characteristic, after completing away from the characteristic of the base field.

Theorem 2.21. *(Grothendieck) Let $\overline{X}$ be an integral proper normal curve of genus g over an algebraically closed field $\overline{k}$ of characteristic $p > 0$, let $X = \overline{X} \setminus \{x_1, ..., x_n\}$ be an open subscheme, and let $\overline{x} : \mathbf{Spec}(\Omega) \to X$ be a geometric point. Then the maximal p'-quotient of $\widehat{\pi}_1(X, \overline{x})$ is isomorphic to the pro-p'-completion of $\Pi_{g,n}$.*

$$\widehat{\pi}_1(X, \overline{x})^{(p')} \simeq \widehat{\Pi}^{(p')}_{g,n}$$

Proof. We will give a brief sketch. Let A be a complete discrete valuation ring of mixed characteristic with algebraically closed fraction field $\overline{K}$ and residue field $\overline{k}$, and let $\eta : \mathbf{Spec}(\overline{K}) \to \mathbf{Spec}(A)$ and

$s : \mathbf{Spec}(\overline{k}) \to \mathbf{Spec}(A)$ denote the generic and closed points of $\mathbf{Spec}(A)$ respectively. Let $\overline{\eta}$ and $\overline{s}$ be geometric points over η and s respectively. Then for $X \to \mathbf{Spec}(A)$ a proper morphism, the induced morphism $\widehat{\pi}_1(X_s, \overline{y}) \xrightarrow{\sim} \widehat{\pi}_1(X, \overline{y})$ is an isomorphism, and if moreover $X \to \mathbf{Spec}(A)$ is flat with X_η and X_s reduced, the induced morphism $\widehat{\pi}_1(X_{\overline{\eta}}, \overline{x}) \to \widehat{\pi}_1(X, \overline{x})$ is surjective. Now for $\overline{s} = s$ consider the specialization map $\mathbf{sp} : \widehat{\pi}_1(X_{\overline{\eta}}, \overline{x}) \to \widehat{\pi}_1(X_{\overline{s}}, \overline{y})$ given by the composition

$$\mathbf{sp} : \widehat{\pi}_1(X_{\overline{\eta}}, \overline{x}) \to \widehat{\pi}_1(X, \overline{x}) \xrightarrow{\sim} \widehat{\pi}_1(X, \overline{y}) \xrightarrow{\sim} \widehat{\pi}_1(X_{\overline{s}}, \overline{y})$$

Then for $X \to \mathbf{Spec}(A)$ a smooth proper morphism with geometrically connected fibers, the specialization map yields an isomorphism $\widehat{\pi}_1(X_{\overline{\eta}}, \overline{x})^{(p')} \simeq \widehat{\pi}_1(X_{\overline{s}}, \overline{y})^{(p')}$.

Now since $X \to \mathbf{Spec}(\overline{k})$ is a smooth curve there exists a compatible system of liftings $X_n \to \mathbf{Spec}(A/\mathfrak{m}^n)$ of X yielding a formal scheme $\mathfrak{X} \to \mathbf{Spec}(A)$ with $X = \mathfrak{X} \times_A \overline{k} \to \mathbf{Spec}(\overline{k})$ proper. Moreover there exists a compatible system of line bundles $\mathcal{L}_n$ on each $X_n = \mathfrak{X} \times_A A/\mathfrak{m}^n$ of $\mathcal{L}$ very ample on X, so by Grothendieck's algebrization theorem [EGAIII] it follows that $\mathfrak{X} \to \mathbf{Spf}(A)$ is obtained from a proper morphism of schemes $\mathcal{X} \to \mathbf{Spec}(A)$ with $X = \mathcal{X} \times_A \overline{k}$. But then $\mathcal{X} \to \mathbf{Spec}(A)$ is smooth since it is smooth on the geometrically connected special fiber $\mathcal{X}_{\overline{s}} = X$, and the generic fiber $\mathcal{X}_{\overline{\eta}}$ is an integral proper normal curve over $\overline{K}$ of characteristic 0, whence the isomorphism

$$\widehat{\Pi}_{g,n}^{(p')} \simeq \widehat{\pi}_1(\mathcal{X}_{\overline{\eta}}, \overline{x})^{(p')} \simeq \widehat{\pi}_1(\mathcal{X}_{\overline{s}}, \overline{y})^{(p')} \simeq \widehat{\pi}_1(X, \overline{y})^{(p')}$$ □

This is nicely illustrated in the case of elliptic curves where the étale fundamental group can be understood as the Tate module, and Grothendieck's comparison theorem can be understood as a statement about the ranks of ℓ-adic Tate modules:

Example 2.12. *Let A be an Abelian variety of dimension g over an algebraically closed field $\overline{k}$ of characteristic $p \geq 0$. For $\varphi : X \to A$ a connected finite étale cover we can equip X with the structure of an Abelian variety over $\overline{k}$ so that φ is an isogeny of Abelian varieties, that is a surjective homomorphism of group schemes with finite kernel, see [Milne] or [Polishchuk].*

We claim that the system of isogenies given by multiplication maps forms a cofinal system in $\mathbf{FEt}_A$ from which we can compute $\widehat{\pi}_1(A, 0)$. On one hand for n prime to p the multiplication map $n_A : A \to A$ is a finite étale Galois cover with group $\mathbf{ker}(n_A)(\overline{k}) \simeq A(\overline{k})[n]$ the n-torsion points of A, and every finite étale cover $\varphi : X \to A$ of degree dividing n arises as a quotient of this cover. On the other hand for n divisible by p the map $n : A \to A$ is no longer finite étale since the group scheme $\mathbf{ker}(n_A)$ is not reduced. Instead consider the action of $A(\overline{k})[n]$ on A by translation; since $A/A(\overline{k})[n] \simeq A$ we obtain a finite étale Galois cover $n'_A : A \to A$ with group $A(\overline{k})[n]$ such that $n_A : n'_A \circ \psi$ for some $\psi : A \to A$ inducing a purely inseparable extension of function fields, and every finite étale cover $\varphi : X \to A$ of degree dividing n arises as a quotient of this cover. It follows that for all $n > 0$ we have $\widehat{\pi}_1(A, 0)/n \simeq A(\overline{k})[n]$ and hence by taking the inverse limit n we obtain

$$\widehat{\pi}_1(A, 0) \simeq \varprojlim_{n \geq 1} \widehat{\pi}_1(A, 0)/n \simeq \varprojlim_{n \geq 1} A(\overline{k})[n] = T(E) \simeq \prod_{\ell} T_\ell(A)$$

Now if $p = 0$ the associated complex analytic space $A^{\mathbf{an}}$ is a complex torus of dimension g and has fundamental group $\pi_1(A^{\mathbf{an}}, 0) \simeq \mathbb{Z}^{2g}$, and hence we have an isomorphism $\widehat{\pi}_1(A, 0) \simeq \widehat{\mathbb{Z}}^{2g}$; in particular each ℓ-adic Tate module $T_\ell(A) \simeq \mathbb{Z}_\ell^{2g}$ has full rank. On the other hand if $p > 0$ we have $T_\ell(A) \simeq \mathbb{Z}_\ell^{2g}$ for $\ell \neq p$ and we have $T_p(A) \simeq \mathbb{Z}_p^\rho$ for $0 \leq \rho \leq g$ the p-rank of A; in other words $\widehat{\pi}_1(A, 0)^{(p')} \simeq (\widehat{\mathbb{Z}}^{2g})^{(p')}$ and $\widehat{\pi}_1(A, 0)^{(p)} \simeq \mathbb{Z}_p^\rho$ is a free Abelian pro-p group.

For instance if E is an elliptic curve over $\overline{k}$ of characteristic $p > 0$, then $\widehat{\pi}_1(E,0)^{(p')} \simeq (\widehat{\mathbb{Z}}^2)^{(p')}$ and $\widehat{\pi}_1(E,0)^{(p)} \simeq \mathbb{Z}_p^{\rho}$ for $0 \leq \rho \leq 1$ the p-rank of E. In particular if $\rho = 0$ then E is called supersingular, as its group of p^n-torsion points is trivial.

Indeed the Tate modules of Abelian varieties in positive characteristic are typical of étale fundamental groups of schemes in positive characteristic: outside the residue characteristic the étale fundamental group agrees with the étale fundamental group for a lift to chraracteristic 0, while at the residue characteristic the étale fundamental group is given in terms of some p-rank. In fact this even holds for integral normal curves in positive characteristic: outside the residue characteristic the étale fundamental group agrees with the étale fundamental group of such a curve in characteristic 0 by Grothendieck's specialization theorem, whereas a theorem of Shafarechich shows at the residue characteristic the étale fundamental group is given in terms of the p-rank of the Jacobian when the curve is proper, and is given as a free pro-p-group generated by wild ramification otherwise.

Theorem 2.22. *(Shafarevich) Let X be an integral normal curve over an algebraically closed field $\overline{k}$ of characteristic $p > 0$. Then the maximal pro-p-quotient $\widehat{\pi}_1(X,\overline{x})^{(p)}$ is a free pro-p-group, of finite rank equal to the p-rank of $\mathbf{Pic}_0(X)$ if X is proper, and of infinite rank equal to the cardinality of $\overline{k}$ if X is affine.*

An immediate consequence of these comparison theorems reveals the structure of étale fundamental groups of integral normal curves over algebraically closed fields:

Example 2.13. *Let X be an integral proper normal curve over an algebraically closed field of characteristic $p \geq 0$. On one hand if $p = 0$ then $\widehat{\pi}_1(X_{\overline{k}},\overline{x})$ is completely determined by the genus g of X in which case $\widehat{\pi}_1(X_{\overline{k}},\overline{x}) \simeq \widehat{\Pi}_{g,0}$. On the other hand if $p > 0$ then $\widehat{\pi}_1(X_{\overline{k}},\overline{x})$ is completely determined by the genus g of X and the p rank $0 \leq \rho \leq g$ of $\mathbf{Pic}_0(X)$. In other words the finite covering theory of such integral proper normal curves is determined by a single parameter in characteristic 0, and by two parameters in positive characteristic.*

Another pertinent application of these comparison theorems has to do with the étale fundamental group of a punctured elliptic curve over an algebraically closed field. In this case the maximal Abelian quotient of the étale fundamental group of such a punctured elliptic curve is isomorphic to the Tate mordule of the associated elliptic curve, possibly completed away from the residue characteristic.

Example 2.14. *Let E be an elliptic curve over an algebraically closed field $\overline{k}$ of characteristic $p \geq 0$ with zero section $\mathcal{O} : \mathbf{Spec}(\overline{k}) \to E$, and let $X = E \setminus \mathcal{O}$ be the associated punctured elliptic curve. On one hand if $p = 0$ then we have $\widehat{\pi}_1^{\mathbf{ab}}(X,\overline{x}) \simeq \widehat{\pi}_1^{\mathbf{ab}}(E,\overline{x}) = \widehat{\pi}_1(E,\overline{x}) \simeq T(E)$ since $\widehat{\pi}_1(X,\overline{x}) \simeq \widehat{\Pi}_{1,1} = \langle \alpha,\beta,\gamma \mid [\alpha,\beta]\gamma = 1\rangle^{\wedge}$ so that $[\alpha,\beta]^{\mathbf{ab}}\gamma^{\mathbf{ab}} = \gamma^{\mathbf{ab}} = 1$; in other words every Abelian finite étale cover of the punctured curve X extends to a finite étale cover of E.*

On the other hand for $p > 0$ we have $\widehat{\pi}_1^{\mathbf{ab}}(X,\overline{x})^{(p')} \simeq \widehat{\pi}^{\mathbf{ab}}(E,\overline{x})^{(p')} = \widehat{\pi}_1(E,\overline{x})$, but $\widehat{\pi}_1^{\mathbf{ab}}(E,\overline{x})^{(p)} \simeq \widehat{\pi}_1(E,\overline{x}) \simeq \mathbb{Z}_p^{\rho}$ for $0 \leq \rho \leq 1$ the p-rank of E, whereas $\widehat{\pi}_1^{\mathbf{ab}}(X,\overline{x})$ is a free Abelian pro-p group of infinite rank. In other words every prime to p Abelian finite étale cover of X extends to a finite étale cover of E, but there exists p-primary Abelian finite étale covers of X which ramified over E: in particular if E has p-rank 0 then every p-primary cover of X ramifies over E.

As a final application, we can combine all the above results along with a Bertini-type argument in order to conclude an important structure theorem on the étale fundamental groups of smooth projective schemes over $\overline{k}$: by Grothendieck's comparison theorems and Shafarevich's theorem we see that the étale fundamental group of a smooth proper curve over $\overline{k}$ is finitely generated as a profinite group, and by cutting a general smooth proper scheme over $\overline{k}$ by hyperplanes we can reduce to the case of curves and conclude that the étale fundamental group of such a scheme is finitely generated as well.

Theorem 2.23. *(Grothendieck) Let X be a smooth projective scheme over an algebraically closed field $\overline{k}$, and let $\overline{x} : \mathbf{Spec}(\Omega) \to X$ be a geometric point. Then $\widehat{\pi}_1(X, \overline{x})$ is topologically finitely generated as a profinite group.*

Proof. Let X be a connected smooth closed subscheme of $\mathbb{P}^n_{\overline{k}}$ of dimension $n > 1$, and let $Y \to X$ be a connected finite étale cover. We can find a hyperplane H in $\mathbb{P}^n_{\overline{k}}$ not containing X such that $X \cap H$ is smooth and connected, and such that the fiber product $Y \times_X (X \cap H)$ is connected. Then $Y \times_X (X \cap H) \to X \cap H$ is a finite étale cover and we obtain a surjection $\widehat{\pi}_1(H \cap X, \overline{x}) \to \widehat{\pi}_1(X, \overline{x})$ for $\overline{x}$ a geometric point of $X \cap H$. But since $X \cap H$ is smooth and projective of dimension strictly less than that of X, we obtain by induction on the dimension of X a surjection $\widehat{\pi}_1(Y, \overline{x}) \to \widehat{\pi}_1(X, \overline{x})$ with $Y = X \cap K$ an integral proper normal curve, K a linear subspace of $\mathbb{P}^n_{\overline{k}}$, and $\overline{x}$ a geometric point of $X \cap K$. Since $\widehat{\pi}_1(Y, \overline{x})$ is topologically finitely generated it follows that $\widehat{\pi}_1(Y, \overline{x})$ is topologically finitely generated as claimed. □

2.4 Etale Homotopy Exact Sequence

To the extent that étale fundamental groups behave very similarly to fundamental groups of topological spaces, we should for instance expect to see an appropriate analog of the homotopy exact sequence in topological groups. Still there are some modifications to the homotopy exact sequence in order for this to work for étale fundamental groups: to begin with, we have only defined étale fundamental groups for connected schemes, so our étale homotopy exact sequence cannot involve the connected components of the schemes involved; moreover, it is not clear what the appropriate notion of fibration yielding a homotopy exact sequence should be, so we must be more specific about the classes of morphisms we work with.

One instance of such an étale homotopy sequence arises from those morphisms of schemes which are flat and proper with geometrically connected fibers. We will only state the result:

Theorem 2.24. *Let $f : X \to S$ be a flat proper morphism of schemes of finite presentation whose geometric fibers are connected and reduced. Suppose that S is connected and let $\overline{s} : \mathbf{Spec}(\Omega) \to S$ be a geometric point, and $\overline{x} : \mathbf{Spec}(\Omega) \to X$ a geometric point over $\overline{s}$. Then we have an exact sequence of étale homotopy groups*

$$\widehat{\pi}_1(X_{\overline{s}}, \overline{s}) \to \widehat{\pi}_1(X, \overline{x}) \to \widehat{\pi}_1(S, \overline{s}) \to 0$$

So far our examples of étale fundamental group have largely fallen into two types: those of non-separably closed fields or similar arithmetic schemes, and those of algebraic varieties over separably closed fields. In the general case of algebraic varieties over non-separably closed fields, we should expect to see both types of finite étale covers. Namely, for X a variety over a field k, we can obtain finite étale covers of X either as geometrically trivial covers $X_L \to X_k$ induced by base change along a finite extension L/k, or as geometric covers $Y_{\overline{k}} \to X_{\overline{k}}$ which happen to be defined over k. Indeed there is a more specific version of the above exact sequence, specifically when $S = \mathbf{Spec}(k)$ for k a field and $X \to \mathbf{Spec}(k)$ a geometrically connected quasi-compact quasi-separated scheme over k, so that the geometric fiber over the unique geometric point $s : \mathbf{Spec}(\overline{k}) \to \mathbf{Spec}(k)$ is the base change $X_{\overline{k}}$. Indeed $X \to \mathbf{Spec}(k)$ does not need to be proper in this case, and we will obtain a short exact sequence of étale fundamental groups which is precisely the étale homotopy sequence for this base change.

We will use the following lemma which amounts to showing exactness at $\widehat{\pi}_1(X_{\overline{k}}, \overline{x})$ which is the hardest step. In particular we only need to show this in the case where k is a perfect field, as the general case follows from the canonical isomorphism $\widehat{\pi}_1(X_{k^{\mathbf{p}}}, \overline{x}) \simeq \widehat{\pi}_1(X, \overline{x})$ when X is geometrically connected. We proceed as follows:

Lemma 2.25. *Let X be a geometrically connected quasi-compact quasi-separated scheme over a perfect field k with fixed algebraic closure $\overline{k}$. Then the canonical morphism $\widehat{\pi}_1(X_{\overline{k}}, \overline{x}) \to \widehat{\pi}_1(X, \overline{x})$ is injective.*

Proof. Let $U \to X$ be a finite étale morphism with U connected so that $U \times_X X_{\overline{k}} \simeq U_{\overline{k}}$. Suppose that the induced morphism $U_{\overline{k}} \to X_{\overline{k}}$ has a section $s : X_{\overline{k}} \to U_{\overline{k}}$; then $s(X_{\overline{k}})$ is an open connected component of $U_{\overline{k}}$ since X is geometrically connected. Now for $\sigma \in \mathbf{Gal}(\overline{k}/k)$ let $\sigma(s) : X_{\overline{k}} \to U_{\overline{k}}$ be the base change of s along σ. Since $U_{\overline{k}} \to X_{\overline{k}}$ is finite étale it has only finitely many sections, in which case the union $\overline{V} = \bigcup_{\sigma \in \mathbf{Gal}(\overline{k}/k)} \sigma(s)(X_{\overline{k}})$ is a finite union and hence is a clopen subset of $U_{\overline{k}}$ which is stable under the canonical $\mathbf{Gal}(\overline{k}/k)$-action. But then $\overline{V}$ is the inverse image of a closed subset V of U, and since the canonical morphism $U_{\overline{k}} \to U$ is open it follows that V is open. But then since U is connected it follows that $U = V$, in which case $U_{\overline{k}}$ is a finite disjoint union of copies of $X_{\overline{k}}$. Moreover the image of $\widehat{\pi}_1(X_{\overline{k}}, \overline{x}) \to \widehat{\pi}_1(X, \overline{x})$ is normal.

Now let $Y \to X_{\overline{k}}$ be a finite étale cover. Then there exists a finite separable extension K/k and a finite étale morphism $U \to X_K$ such that $Y = X_{\overline{k}} \times_{X_K} U = U \times_K \overline{k}$. But then the composition $U \to X_K \to X$ is finite étale and $U_{\overline{k}}$ contains $Y = U \times_K \overline{k}$ as an open and closed subscheme. □

Theorem 2.26. *Let k be a field with fixed separable closure $k^{\mathbf{s}}$ and fixed algebraic closure $\overline{k}$, and let X be a geometrically connected quasi-compact quasi-separated scheme over k. Then for $\overline{x}$ a geometric point of X we have a short exact sequence of profinite groups*

$$0 \to \widehat{\pi}_1(X_{\overline{k}}, \overline{x}) \to \widehat{\pi}_1(X, \overline{x}) \to \mathbf{Gal}(k^{\mathbf{s}}/k) \to 0$$

Proof. We may assume that k is a perfect field since for $k^{\mathbf{p}}$ its perfection and for X is connected over k then $X_{k^{\mathbf{p}}}$ is connected, and since the canonical morphism $\mathbf{Spec}(k^{\mathbf{p}}) \to \mathbf{Spec}(k)$ is a universal homeomorphism its base change induces an isomorphism $\widehat{\pi}_1(X_{k^{\mathbf{p}}}, \overline{x}) \simeq \widehat{\pi}_1(X, \overline{x})$.

By the previous lemma we have shown that the above short sequence is exact at $\widehat{\pi}_1(X_{\overline{k}}, \overline{x})$, so it remains to show exactness at $\widehat{\pi}_1(X, \overline{x})$ and $\mathbf{Gal}(\overline{k}/k)$. On one hand recall that connected objects of $\mathbf{FEt}_k$ are of the form $\mathbf{Spec}(K) \to \mathbf{Spec}(k)$ with K/k a finite separable extension. Then if X_K is connected then the canonical morphism $X_{\overline{k}} \to X_K$ is surjective since $X_{\overline{k}}$ is conencted by assumption, in which case the induced morphism $\widehat{\pi}_1(X, \overline{x}) \to \mathbf{Gal}(\overline{k}/k)$ is surjective and hence the above short sequence is exact at $\mathbf{Gal}(\overline{k}/k)$.

On the other hand it is clear that the composition $\mathbf{FEt}_k \to \mathbf{FEt}_k \to \mathbf{FEt}_{X_{\overline{k}}}$ induced by base change sends objects to disjoint unions of copies of $X_{\overline{k}}$, since schemes finite étale over $\mathbf{Spec}(k)$ are of the form $\mathbf{Spec}(K_1) \amalg ... \amalg \mathbf{Spec}(K_n)$ with each K_i/k a finite separable extension, each of which is sent by base change to the trivial cover in $\mathbf{FEt}_{X_{\overline{k}}}$. Consequently the induced composition $\widehat{\pi}_1(X_{\overline{k}}, \overline{x}) \to \widehat{\pi}_1(X, \overline{x}) \to \mathbf{Gal}(\overline{k}/k)$ is trivial and hence the above short sequence is exact at $\widehat{\pi}_1(X, \overline{x})$. The result follows. □

Example 2.15. *Let X be a normal scheme over a field k with fixed algebraic closure $\overline{k}$. We have seen that for X an integral normal scheme with function field $K(X)$, and for $M = K(X)^{\mathbf{ur}}$ a maximal unramified extension of $K(X)$, we have an isomorphisms $\widehat{\pi}_1(X,\overline{x}) \simeq \mathbf{Gal}(M/K(X))$ and $\widehat{\pi}_1(X_{\overline{k}},\overline{x}) \simeq \mathbf{Gal}(M/K(X_{\overline{k}}))$, in which case we have an isomorphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\pi}_1(X_{\overline{k}},\overline{x}) & \longrightarrow & \widehat{\pi}_1(X,\overline{x}) & \longrightarrow & \mathbf{Gal}(\overline{k}/k) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathbf{Gal}(M/K(X_{\overline{k}})) & \longrightarrow & \mathbf{Gal}(M/K(X)) & \longrightarrow & \mathbf{Gal}(K(X_{\overline{x}})//K(X)) & \longrightarrow & 0 \end{array}$$

Indeed this exact sequence is just the exact sequence exhibiting the quotient of Galois groups

$$\mathbf{Gal}(M/K(X))/\mathbf{Gal}(M/K(\overline{X})) \simeq \mathbf{Gal}(K(\overline{X})/K(X)) \simeq \mathbf{Gal}(\overline{k}/k)$$

Now the action of $\widehat{\pi}_1(X,\overline{x})$ on itself by conjugation restricts to an action on $\widehat{\pi}_1(\overline{X},\overline{x})$, and hence yields a homomorphism $\widehat{\pi}_1(X,\overline{x}) \to \mathbf{Aut}(\widehat{\pi}_1(\overline{X},\overline{x}))$. But then the elements of $\widehat{\pi}_1(\overline{X},\overline{x}) \hookrightarrow \widehat{\pi}_1(X,\overline{x})$ act by inner automorphism on $\widehat{\pi}_1(\overline{X},\overline{x})$ and by passing to the quotient we have the outer Galois representation $\mathbf{Gal}(k^{\mathbf{s}}/k) \to \mathbf{Out}(\widehat{\pi}_1(\overline{X},\overline{x}))$ which can be expressed as a morphism of short exact sequences of profinite groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\pi}_1(\overline{X},\overline{x}) & \longrightarrow & \widehat{\pi}_1(X,\overline{x}) & \longrightarrow & \mathbf{Gal}(k^{\mathbf{s}}/k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Inn}(\widehat{\pi}_1(\overline{X},\overline{x})) & \longrightarrow & \mathbf{Aut}(\widehat{\pi}_1(\overline{X},\overline{x})) & \longrightarrow & \mathbf{Out}(\widehat{\pi}_1(\overline{X},\overline{x})) & \longrightarrow & 0 \end{array}$$

If the étale homotopy exact sequence splits with a section $s : \mathbf{Gal}(k^{\mathbf{s}}/k) \to \widehat{\pi}_1(X,\overline{x})$, then the outer Galois representation $\rho : \mathbf{Gal}(k^{\mathbf{s}}/k) \to \mathbf{Out}(\widehat{\pi}_1(\overline{X},\overline{x}))$ lifts to a Galois representation $\rho : \mathbf{Gal}(k^{\mathbf{s}}/k) \to \mathbf{Aut}(\widehat{\pi}_1(\overline{X},\overline{x}))$.

Example 2.16. *Let $X = \mathbb{P}^1_k \setminus \{0,\infty\}$ over a number field k and let $\overline{x} : \mathbf{Spec}(\Omega) \to X$ be a geometric point. We have seen that $\widehat{\pi}_1(X_{\overline{k}},\overline{x}) \simeq \widehat{\mathbb{Z}}$, so we have the étale homotopy exact sequence*

$$0 \to \widehat{\mathbb{Z}} \to \widehat{\pi}_1(X,\overline{x}) \to \mathbf{Gal}(\overline{k}/k) \to 0$$

Since $\widehat{\pi}_1(X_{\overline{k}},\overline{x}) \simeq \widehat{\mathbb{Z}}$ is Abelian the outer Galois representation $\rho_{out} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Out}(\widehat{\pi}_1(X_{\overline{k}},\overline{x}))$ lifts canonically to a Galois representation $\rho : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{k}},\overline{x}))$.

Now for $n > 0$ we have cyclic quotients $\widehat{\pi}_1(X_{\overline{k}},\overline{x})/n \simeq \mathbb{Z}/n$ identified with $\mathbf{Gal}(K_n/\overline{k}(t))$ where K_n is the splitting field of $x^n - t \in \overline{k}(t)[x]$ corresponding to the cyclic Galois cover of $X_{\overline{k}}$. Then we have a generator $\varphi_n \in \mathbf{Gal}(K_n/\overline{k}(t))$ sending a fixed n-th root $\sqrt[n]{t} \in K_n$ to $\zeta_n \sqrt[n]{t}$ for $\zeta_n \in \overline{k}$ a fixed primitive n-th root of unity. Then $\sigma \in \mathbf{Gal}(\overline{k}/k)$ acts in $\mathbf{Gal}(K_n/\overline{k}(t))$ by sending φ_n to the automorphism $\sigma\cdot\varphi_n$ sending $\sqrt[n]{t}$ to $\sigma(\zeta_n)\sqrt[n]{t} = \zeta_n^{\chi_n(\sigma)}\sqrt[n]{t}$ where $\chi_n(\sigma) \in (\mathbb{Z}/n)^\times$ is the cyclotomic character of σ. Now there Galois actions are compatible since a choice of isomorphism $\widehat{\pi}_1(X_{\overline{k}},\overline{x}) \simeq \mathbf{Gal}(\bigcup_{n\geq 1} K_n/\overline{k}(t))$ determines a compatible system $(\zeta_n)_{n\geq 1}$ of primitive n-th roots of unity with $\zeta_{nm}^m = \zeta_n$ for $n \mid m$; then $\sigma \in \mathbf{Gal}(\overline{k}/k)$ acts on the compatible system $(\zeta_n)_{n\geq 1}$ by $\sigma\cdot(\zeta_n)_{n\geq 1} = (\sigma\cdot\zeta_n)_{n\geq 1} = (\zeta_n^{\chi_n(\sigma)})_{n\geq 1} = (\zeta_n)_{n\geq 1}^{\chi(\sigma)}$. It follows that the Galois representation $\rho : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(\widehat{\mathbb{Z}}) \simeq \widehat{\mathbb{Z}}^\times$ is equivalently the cyclotomic character $\chi : \mathbf{Gal}(\overline{k}/k) \to \widehat{\mathbb{Z}}^\times$.

In other words the cyclotomic character $\chi : \mathbf{Gal}(\overline{k}/k) \to \widehat{\mathbb{Z}}^\times$ governs the action of $\mathbf{Gal}(\overline{k}/k)$ on finite étale covers of $X = \mathbb{P}^1_{\overline{k}} \setminus \{0, \infty\}$, which is completely determined by the action on roots of unity in each of the cyclic Galois covers of X.

Another nice example which we will not explore in too much detail is the Galois representation on the étale fundamental group of an Abelian variety; in this case the outer Galois representation lifts canonically to a Galois representation in the Tate module of the Abelian variety, and indeed this is the usual Galois representation one encounters in the theory of Abelian varieties.

Example 2.17. *Let A be an Abelian variety of dimension g over a number field k with zero section $\mathcal{O} : \mathbf{Spec}(k) \to A$. We have seen that $\widehat{\pi}_1(A_{\overline{k}}, \overline{x})$ is isomorphic to the Tate module $T(A) \simeq \prod_\ell T_\ell(A)$, so we have the étale homotopy exact sequence with canonical choice of section $s_{\mathcal{O}} : \mathbf{Gal}(\overline{k}/k) \to \widehat{\pi}_1(A, \mathcal{O})$*

$$0 \to T(A) \to \widehat{\pi}_1(A, \mathcal{O}) \to \mathbf{Gal}(\overline{k}/k) \to 0$$

Now the outer Galois representation $\rho_{out} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Out}(T(A))$ lifts canonically [2] to a Galois representation $\rho : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(T(A))$ which yields for each prime ℓ of $\mathbb{Q}$ a Galois representation $\rho_\ell : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(T_\ell(A))$ on the ℓ-adic Tate module of A, with determinant $\mathbf{det}\rho_\ell : \mathbf{Gal}(\overline{k}/k) \to \mathbb{Z}_\ell^\times$ the ℓ-adic cyclotomic character [Serre].

The Galois representation $\rho : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(T(A))$ famously contains a lot of information about the arithmetic of A. For instance by [Neron-Ogg-Shafarevich] A has good reduction at a prime $\mathfrak{p}$ of k precisely if for some prime ℓ of $\mathbb{Q}$ not dividing $\mathfrak{p}$ the ℓ-adic Galois representation $\rho_\ell : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(T_\ell(A)) \simeq \mathbf{GL}_{2g}(\mathbb{Z}_\ell)$ is unramified at $\mathfrak{p}$, that is its kernel contains the inertia group $I_\mathfrak{p}$. Moreover there are several criteria by Serre which says when the ℓ-adic Galois representation $\rho_\ell : \mathbf{Gal}(\overline{k}/k) \to \mathbf{GL}_2(\mathbb{Z}_\ell)$ is surjective.

Example 2.18. *Let $X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ and let $\overline{x} : \overline{k} \to X$ be a geometric point. We know that $X^{\mathbf{an}}$ has fundamental group F_2 generated by loops x and y around 0 and 1 respectively, and hence $\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \overline{x}) \simeq \pi_1(X, \overline{x})^\wedge \simeq \widehat{F}_2$. Then we have the étale homotopy exact sequence*

$$0 \to \widehat{F}_2 \to \widehat{\pi}_1(X, \overline{x}) \to \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 0$$

Since $\widehat{\pi}_1(X_{\overline{k}}, \overline{x}) \simeq \widehat{F}_2$ is not Abelian, we no longer have a canonical choice of lifting of the outer Galois representation $\rho_{out} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Out}(\widehat{F}_2)$ to a Galois representation $\rho : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(\widehat{F}_2)$; indeed any choice of k-rational point $x : \mathbf{Spec}(k) \to X$ yields a particular Galois representation $\rho_x : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(\widehat{F}_2)$, but this depends on the choice of basepoint x and is not easily calculated.

Indeed we need a more refined approach in order to calculate the above Galois representation associated to the étale fundamental group of $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$; the problem is that we need a canonical choice of basepoints for $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ which are invariant under the automorphisms of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$, but the only basepoints for which this is the case are the punctures $\{0, 1, \infty\}$. This is precisely what is accomplished with tangential basepoints, which after some digression will yield an explicit formula for the Galois representation mentioned in the introduction.

[2] Put another way, the choice of zero section $\mathcal{O} : \mathbf{Spec}(k) \to A$ yields a section $\rho_{\mathcal{O}} : \mathbf{Gal}(\overline{k}/k) \to \widehat{\pi}_1(A, \mathcal{O})$ of the étale homotopy exact sequence, and since $\widehat{\pi}_1(A_{\overline{k}}, \mathcal{O})$ is Abelian the resulting Galois representation $\rho_{\mathcal{O}} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{GL}_{2g}(\widehat{\mathbb{Z}})$ does not depend on the choice of splitting.

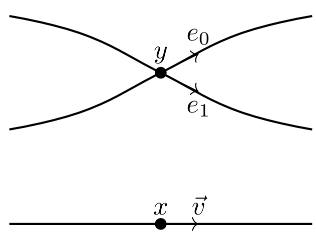
2.5 Tangential Basepoints

In certain situations it is useful to consider fundamental groups based at a point on some boundary component or at some puncture. The most pertinent example of this is that $X = \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$; although any choice of $\mathbb{Q}$-rational basepoint yields a section of its étale homotopy exact sequence, we would like to take the points $\{0, 1, \infty\}$ as the most natural set of $\mathbb{Q}$-rational basepoints, even though they do not belong to X but rather its compactification $\overline{X} = \mathbb{P}^1_{\overline{\mathbb{Q}}}$. In fact there is more reason to do this if we work over $\mathbb{Z}$: given a point $x \in \mathbb{Z}\setminus\{0, 1\}$, the pair $(\mathbb{P}^1_{\mathbb{Z}}\setminus\{0, 1, \infty\}, x)$ has bad reduction at a prime p whenever $p \mid x(x-1)$ as $x \equiv 0, 1 \mod p$ which is not contained in $\mathbb{P}^1_{\mathbb{F}_p} \setminus \{0, 1, \infty\}$; consequently the pair $(\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}, x)$ has good reduction at every prime p precisely if $x \in \{0, 1\}$, which cannot be the case for any $\mathbb{Q}$-rational basepoint x.

One way to remedy this is to consider so called $\mathbb{Q}$-rational tangential basepoints of X. Let $\pi : Y \to X$ be a finite étale cover, and let $\overline{Y} \to \overline{X}$ be its compactification; although taking the fiber above a point $x \in \{0, 1, \infty\}$ does not define an appropriate fiber functor $F_x : \mathbf{FEt}_X \to \mathbf{Fin}$, taking the fiber above a tangent vector $\vec{v} \in T_x\overline{X}$ does.

Definition 2.7. *Let X be an integral proper normal curve over a field k and let $\overline{X}$ be its compactification. A k-rational tangential basepoint $\vec{v} : \mathbf{Spec}\big(k((t))\big) \to \overline{X}$ of X consists of a k-rational point $x : \mathbf{Spec}(k) \to \overline{X}$ along with a choice of tangent vector $\vec{v} \in T_x\overline{X}$.*

Given such a k-rational tangential basepoint $\vec{v} : \mathbf{Spec}\big(k((t))\big) \to \overline{X}$ and a finite étale cover $Y \to X$ we can consider the fiber of $\overline{Y}$ above the tangent vector $\vec{v} \in T_x\overline{X}$ which not only recovers the usual fiber over points $x \pm \varepsilon$ arbitrarily close to x, but also the ramification indices over x by identifying those tangent vectors $e \in \pi^{-1}(\vec{v})$ which share a common origin. For this reason the fiber over a k-rational tangential basepoint encodes more data than the fiber over a typical k-rational basepoint.



We define $\mathcal{F}_{\vec{v}}(\overline{Y})$ as the set of pairs (y, e) with $y \in \pi^{-1}(x)$ and $e \in \pi^{-1}(\vec{v})$ so that $e \in T_y\overline{Y}$. Similarly we define $\mathcal{E}_{\vec{v}}(\overline{Y}) = \#\mathcal{F}_{\vec{v}}(\overline{Y})$ as the cardinality of $\mathcal{F}_{\vec{v}}(\overline{Y})$, necessarily equal to the degree of the finite étale cover $\pi : Y \to X$. We also define $\mathcal{S}_{\vec{v}}$ as the sector obtained from a sufficiently small open disc centered at x with the semi-axis opposite to the tangent vector $\vec{v}$ removed, which we should think of as a domain for analytic functions centered at x with a branch cut opposite to the tangent vector $\vec{v}$. In this situation the assignment of each finite étale cover $Y \to X$ with compactification $\overline{Y} \to \overline{X}$ to the fiber $\mathcal{F}_{\vec{v}}(\overline{Y})$ defines a functor $F_{\vec{v}} : \mathbf{FEt}_X \to \mathbf{Fin}$. We borrow a theorem of Deligne which says that this yields an appropriate fiber functor for the Galois category $\mathbf{FEt}_X$.

Theorem 2.27. *(Deligne) Let X be an integral normal curve over a field k of characteristic 0, and let $\vec{v} : \mathbf{Spec}\big(k((t))\big) \to X$ be a k-rational tangential basepoint. Then $\mathbf{FEt}_X$ is a Galois category with respect to the fiber functor $\mathcal{F}_{\vec{v}} : \mathbf{FEt}_X \to \mathbf{Fin}$.*

Now fix a k-rational (tangential) basepoint $\vec{v}$ of X; in this situation we have a convenient description of the étale fundamental groups $\widehat{\pi}_1(X, \vec{v})$ and $\widehat{\pi}_1(X_{\overline{k}}, \vec{v})$. Namely, let $\mathcal{M}_{\vec{v}}$ be the maximal unramified extension of K, that is the direct limit of those function fields L/K for which the normalization of X in L is unramified.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\pi}_1(X_{\overline{k}}, \vec{v}) & \longrightarrow & \widehat{\pi}_1(X, \vec{v}) & \longrightarrow & \mathbf{Gal}(\overline{k}/k) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathbf{Gal}(\mathcal{M}_{\vec{v}}/\overline{K}) & \longrightarrow & \mathbf{Gal}(\mathcal{M}_{\vec{v}}/K) & \longrightarrow & \mathbf{Gal}(\overline{K}/K) & \longrightarrow & 0 \end{array}$$

Suppose first that $\vec{v}$ is represented by a k-rational point $x : \mathbf{Spec}(k) \to X$ as considered in previous sections, so that by functoriality we obtain a section $s_x : \mathbf{Gal}(\overline{k}/k) \to \widehat{\pi}_1(X, x)$ and the monodromy representation

$$\rho_{X,x} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{k}}, x))$$

given by $\rho_{X,x}(\sigma)(\gamma) = s_x(\sigma)\gamma s_x(\sigma)^{-1}$. Indeed in order to calculate the monodromy representation $\rho_{X,x}$ it suffices to compute the section $s_x(\sigma) \in \mathbf{Gal}(\mathcal{M}_x/K)$ for each $\sigma \in \mathbf{Gal}(\overline{k}/k)$. To that end let $Y \to X$ be a finite étale cover and let $Y(\mathbb{C}) \to X(\mathbb{C})$ be the induced cover. Suppose that Y has function field L a finite unramified extension of K. We associate to each point $y \in \pi^{-1}(x)$ an embedding $\varphi : L \hookrightarrow \mathbb{C}((t-x))$ as follows: each $f \in L$ can be viewed locally as an analytic function in $t - x$ through the local isomorphism $\pi : Y(\mathbb{C}) \to X(\mathbb{C})$ which is uniquely determined by its Laurent series expansion

$$f(t) = \sum_{i \geq -n} a_i (t-x)^i \in \mathbb{C}((t-x))$$

In fact each such f is contained in the field of convergent Laurent series $\mathbb{C}((t-x))_0$ of which $\mathbb{C}((t-x))$ is a completion. Since f is uniquely determined by its Laurent series expansion at y it follows that the choice of $y \in \pi^{-1}(x)$ yields an embedding $\varphi : L \hookrightarrow \mathbb{C}((t-x))$. In fact this construction is algebraic: as Y is defined over $\overline{k}$ we have $a_i \in \overline{k}$ in the above Laurent series expansion of f so that $\varphi(f) \in \overline{k}((t-x))_0 \hookrightarrow \overline{k}((t-x))$.

Now the topological fundamental group $\pi_1(X(\mathbb{C}), x)$ acts on this set of embeddings as follows: for $\gamma \in \pi_1(X(\mathbb{C}), x)$ a loop and $f \in L$ a rational function, and given a choice of embedding $\varphi : L \hookrightarrow \mathbb{C}((t-x))$ as above, we define the monodromy action of γ on f by

$$\gamma \cdot \varphi(f) = \varphi(\gamma^{-1} \cdot f) \in \mathbb{C}((t-x))$$

where $\gamma^{-1} \cdot f$ is the analytic continuation of f along the loop γ. Indeed the action of γ on the set $\mathcal{F}_x(Y)$ is identical to the monodromy action of γ on the fiber F_x over x. Passing to the inverse limit over finite étale covers $Y \to X$ we obtain:

Theorem 2.28. *To every coherent family of places of $\mathcal{M}_x$ over $x : \mathbf{Spec}(k) \to X$ is associated an embedding $\varphi : \mathcal{M}_x \hookrightarrow \overline{k}((t-x))$ such that the image of any finite extension $L \in \mathbf{FEt}_{K(X)}$ is contained in the subfield $\overline{k}((t-x))_0 \hookrightarrow \overline{k}((t-x))$ of convergent Laurent series. Moreover each $\gamma \in \widehat{\pi}_1(X_{\overline{k}}, x)$ acts on the set $\mathcal{F}_x$ of such embeddings.*

Suppose now that $\vec{v}$ is represented by a k-rational tangential basepoint, that is a morphism $\vec{v} : \mathbf{Spec}\big(k((t))\big) \to \overline{X}$ or equivalently a k-rational basepoint $x : \mathbf{Spec}(k) \to \overline{X}$ along with a choice of tangent vector $\vec{v} \in T_x\overline{X}$. Now for $\pi : Y \to X$ a finite étale cover with function field F over $\overline{k}$ and compactification $\pi : \overline{Y} \to \overline{X}$, each element of $\mathcal{F}_{\vec{v}}(\overline{Y})$ determines an embedding into the field $\overline{k}\{\{t_{\vec{v}}\}\}$ into the field of Puiseux series over $\overline{k}$

$$\varphi : F \hookrightarrow \overline{k}\{\{t_{\vec{v}}\}\} = \bigcup_{N\geq 1} \overline{k}((t_{\vec{v}}))$$

Indeed choose a local coordinate t of $\overline{Y}$ centered at y such that the induced cover $\pi : \overline{Y}(\mathbb{C}) \to \overline{X}(\mathbb{C})$ is locally isomorphic to the cyclic cover $\pi(t) = t^N$ centered at x. Then the choice of tangent vector $\vec{w} \in \pi^{-1}(\vec{v}) \hookrightarrow T_y\overline{Y}$ determines a uniformizing parameter of t such that its restriction to the edge determined by $\vec{w}$ is positive real. In this situation, each $f \in F$ admits a convergent Laurent series expansion in z centered at y

$$f(z) = \sum_{i\geq -n} a_i t^i$$

with $n \geq 0$ and $a_i \in \overline{k}$. Since the induced cover $\pi : \overline{Y}(\mathbb{C}) \to \overline{X}(\mathbb{C})$ is locally isomorphic to the cyclic cover $\pi(t) = t^N$ centered at x, we define the convergent Puiseux series expansion associated to f

$$\widetilde{f}(t_{\vec{v}}) = \sum_{i\geq -n} a_i t_{\vec{v}}^{i/N}$$

The assignment of each $f \in F$ to its Puiseux series expansion $\widetilde{f} \in \overline{k}\{\{t_{\vec{v}}\}\}$ determines an embedding $\varphi : F \hookrightarrow \overline{k}\{\{t_{\vec{v}}\}\}$. More precisely, any $\widetilde{f}$ associated to f is convergent as a Puiseux series and hence defines a germ of meromorphic function $\widetilde{f} \in \mathcal{P}_{\vec{v}}(\overline{k}(t_{\vec{v}})) \subseteq \mathcal{M}_{\vec{v}}$ on $\mathcal{S}_{\vec{v}}$. Indeed $\mathcal{P}_{\vec{v}}(\overline{k}(t_{\vec{v}}))$ is the field of convergent Puiseux series with coefficients in $\overline{k}$, of which the field $\overline{k}\{\{t_{\vec{v}}\}\}$ is a completion. Consequently each element of $\mathcal{F}_{\vec{v}}(\overline{Y})$ defines an embedding $\varphi : F \hookrightarrow \mathcal{P}_{\vec{v}}(\overline{k}(t_{\vec{v}})) \hookrightarrow \overline{k}\{\{t_{\vec{v}}\}\}$, and conversely any place of F above $\vec{v}$ determines a point $y_0 \in \pi^{-1}(0)$ and a uniformizing parameter z, that is the pullback of $t_{\vec{v}}^{1/N}$ in the completion of F at the place y_0; then the uniformizing parameter determines an edge in Y of type $\vec{v}$, and hence an element of $\mathcal{F}_{\vec{v}}(\overline{Y})$ by requiring the uniformizing parameter z to be real positive.

Theorem 2.29. *To ever coherent family of places of $\mathcal{M}_{\vec{v}}$ over $\vec{v} : \mathbf{Spec}\big(k((t))\big) \to X$ is associated an embedding $\varphi : \mathcal{M}_{\vec{v}} \hookrightarrow \overline{k}\{\{t_{\vec{v}}\}\}$ such that the image any finite extension $F \in \mathbf{FEt}_{K(X)}$ is contained in the subfield $\mathcal{P}_{\vec{v}}(\overline{k}(t_{\vec{v}})) = \overline{k}\{\{t_{\vec{v}}\}\}_0 \hookrightarrow \overline{k}\{\{t_{\vec{v}}\}\}$ of convergent Puiseux series, yielding an embedding*

$$\varphi : F \hookrightarrow \mathcal{P}_{\vec{v}}(\overline{k}(t_{\vec{v}})) \hookrightarrow \overline{k}\{\{t_{\vec{v}}\}\}$$

Now we want to describe the Galois representation $\rho_{\vec{v}} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{k}}, \vec{v}))$ on the étale fundamental group induced by the k-rational tangential basepoint $\vec{v}$. We will compute this as follows: we will first describe the action of each $\gamma \in \widehat{\pi}_1(X_{\overline{k}}, \vec{v})$ on Puiseux series expansions $\varphi(f) \in \overline{k}\{\{t_{\vec{v}}\}\}$, and then we will describe how each $\sigma \in \mathbf{Gal}(\overline{k}/k)$ conjugates this action and hence defines a Galois representation by comparison of Puisseux coefficients.

To that end let $\pi : Y \to X$ be a finite étale cover with function field F over $\overline{\mathbb{Q}}(t)$, and let $\varphi \in \mathcal{E}_{\vec{v}}$ be an embedding $\varphi : F \hookrightarrow \mathcal{P}_{\vec{v}}(\overline{\mathbb{Q}}(Y)) \hookrightarrow \overline{\mathbb{Q}}\{\{t_{\vec{v}}\}\}$. Let $\gamma \in \widehat{\pi}_1(X^{\mathbf{an}}, \vec{v})$ be a loop based at $\vec{v}$, and let $f \in F$. We define the monodromy action of γ on f by analytic continuation

$$\gamma \cdot \varphi(f) = \varphi(\gamma^{-1} \cdot f) \in \mathcal{P}_{\vec{v}}(\overline{\mathbb{Q}}(Y))$$

In other words $\varphi(f) \in \mathcal{P}_{\vec{v}}(\overline{\mathbb{Q}}(Y)) \subseteq \overline{\mathbb{Q}}\{\{t\}\}$ defines a germ of analytic function on the sector $\mathcal{S}_{\vec{v}}$ which can be analytically continued along the path γ to a germ of analytic section $\gamma \cdot \varphi(f) \in \mathcal{P}_{\vec{w}}(\overline{\mathbb{Q}}(Y))$ on the sector $\mathcal{S}_{\vec{w}}$. Moreover the assignment of each embedding $\varphi \in \mathcal{E}_{\vec{v}}(\overline{k}(Y))$ to the deformed embedding $\gamma \cdot \varphi \in \mathcal{E}_{\vec{w}}(\overline{k}(Y))$ is natural with respect to the finite étale cover $\pi : Y \to X$.

We can define an action of $\mathbf{Gal}(\overline{k}/k)$ on $\widehat{\pi}_1(X_{\overline{k}}, \vec{v})$ as follows: let $\varphi \in \mathcal{E}_{\vec{v}}(\overline{k}(Y))$ and $\gamma \in \widehat{\pi}_1(X_{\overline{k}}, \vec{v})$; then the action of $\sigma \in \mathbf{Gal}(\overline{k}/k)$ is given by the conjugation action $\sigma \cdot \gamma = \sigma\gamma\sigma^{-1}$, that is the element

$$\sigma \cdot \gamma(\varphi) = \sigma \cdot (\gamma \cdot (\sigma^{-1} \cdot \varphi)) \in \mathcal{E}_{\vec{v}}(\overline{k}(Y))$$

In other words for $f \in F$ a test function and $\varphi \in \mathcal{E}_{\vec{v}}(\overline{k}(Y))$ and $\varphi(f) \in \mathcal{P}_{\vec{v}}(\overline{k}(Y))$ its Puiseux series expansion we compute $(\sigma \cdot \gamma)(\varphi)(f) = \sigma \cdot (\gamma \cdot (\sigma^{-1} \cdot \varphi))(f)$ by applying σ^{-1} to the coefficients of $\varphi(f)$, analytically continuing along γ, and applying σ to the coefficients of $\gamma \cdot (\sigma^{-1} \cdot \varphi)(f)$. By comparing the coefficients of $\gamma \cdot \varphi(f)$ and $(\sigma\gamma\sigma^{-1}) \cdot \gamma(\varphi)$ this yields a Galois representation in the étale fundamental group

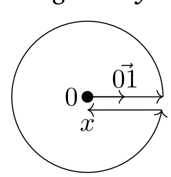
$$\rho_{\vec{v}} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{k}}, \vec{v}))$$

Indeed this provides a lift of the outer Galois representation $\rho_{\text{out}} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Out}(\widehat{\pi}_1(X_{\overline{k}}, \vec{v}))$ associated to the étale homotopy exact sequence of the étale fundamental group $\widehat{\pi}_1(X_{\overline{k}}, \vec{v})$ and $\widehat{\pi}_1(X, \vec{v})$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\pi}_1(X_{\overline{k}}, \vec{v}) & \longrightarrow & \widehat{\pi}_1(X, \vec{v}) & \underset{\curvearrowleft s_{\vec{v}}}{\longrightarrow} & \mathbf{Gal}(\overline{k}/k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \swarrow \rho_{\vec{v}} & \downarrow \rho_{\text{out}} & & \\
0 & \longrightarrow & \mathbf{Inn}(\widehat{\pi}_1(X_{\overline{k}}, \vec{v})) & \longrightarrow & \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{k}}, \vec{v})) & \longrightarrow & \mathbf{Out}(\widehat{\pi}_1(X_{\overline{k}}, \vec{v})) & \longrightarrow & 0
\end{array}$$

It is worth seeing this worked out explicitly since there are so many moving parts; as an example we will show that the Galois representation in the étale fundamental group $\widehat{\pi}_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, \infty\}, \vec{01})$ is the cyclotomic character as expected.

Example 2.19. *Let $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, \infty\}$ with compactification $\overline{X} = \mathbb{P}^1_{\mathbb{Q}}$, and consider a local coordinate $t_{\vec{01}} = t$ centered at 0 yielding an isomorphism between the function field of $X_{\overline{\mathbb{Q}}}$ and $\overline{\mathbb{Q}}(t_{\vec{01}})$. Consider the tangential basepoint $\vec{01}$ associated to the local coordinate $t_{\vec{01}}$; here $\vec{01}$ represents the unit tangent vector $\frac{d}{dt} t_{\vec{01}}|_{t=0}$ based at 0 pointing towards 1, so that the tangential basepoint $\vec{01} : \mathbf{Spec}\big(\mathbb{Q}((t))\big) \to \overline{X}$ is $\mathbb{Q}$-rational with underlying $\mathbb{Q}$-rational basepoint $0 : \mathbf{Spec}(\mathbb{Q}) \to \overline{X}$. In this case the étale fundamental group $\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{01}) \simeq \widehat{\mathbb{Z}}$ is generated by the loop x based at $\vec{01}$, that is the simple closed clockwise loop around 0 in $X^{\mathbf{an}}$ which starts and ends at 0 tangentially to $\vec{01}$.*



We obtain a Galois representation in the étale fundamental group $\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{01}) \simeq \mathbf{Gal}(\mathcal{M}_{\vec{01}}/\overline{\mathbb{Q}}(t_{\vec{01}}))$ where $\mathcal{M}_{\vec{01}}$ is a fixed maximal unramified extension of the function field $\overline{\mathbb{Q}}(t_{\vec{01}})$.

$$\rho_{\vec{01}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{01})) = \widehat{\mathbb{Z}}^{\times}$$

We compute this as follows: let $Y(\overline{\mathbb{Q}}) \to X(\overline{\mathbb{Q}})$ be a finite étale cover which we may assume to be the standard cyclic Galois cover of order N, and let $f \in \mathcal{M}_{\vec{01}}$ be a function so that $\varphi(f) = \sum_{i\geq -n} a_n t_{\vec{01}}^{i/N}$ for $a_i \in \overline{\mathbb{Q}}$. Writing $\varphi(f) = \sum_{i\geq -n} a_i \exp(\frac{i}{N}\log(t_{\vec{01}}))$, the analytic continuation of f along the loop x is given

$$x \cdot \varphi(f) = \sum_{i\geq -n} a_i \exp(\tfrac{i}{N}\log(t_{\vec{01}}) + \tfrac{i}{N}2\pi\sqrt{-1}) = \sum_{i\geq -n} \zeta_N^i \exp(\tfrac{i}{N}\log(t_{\vec{01}}))$$

Then $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $x \cdot \varphi(f)$ by first applying σ^{-1} to the a_i in $\varphi(f)$, then applying x as above, then applying σ to the $\sigma^{-1}(a_i)$ and the other terms. We obtain:

$$(\sigma x \sigma^{-1}) \cdot \varphi(f) = \sum_{i\geq -n} \zeta_N^{i\chi_N(\sigma)} \exp(\tfrac{i}{N}\log(t_{\vec{01}}))$$

By comparing the coefficients of $x \cdot \varphi(f)$ and $(\sigma x \sigma^{-1}) \cdot \varphi(f)$ it follows that σ acts as $\sigma \cdot x = x^{\chi(\sigma)}$, that is on each cyclic Galois cover of degree N and each lift ζ_N of x regarded as a choice of N-th root of unity we have $\sigma \cdot \zeta_N = \zeta_N^{\chi_N(\sigma)}$ for $\chi_N(\sigma) \in (\mathbb{Z}/n\mathbb{Z})^\times$ as required.

We now want to extend the Galois action on the étale fundamental group $\widehat{\pi}_1(X_{\overline{k}}, \vec{v})$ with respect to a tangential basepoint $\vec{v}$ to a Galois action on the étale fundamental groupoid $\widehat{\pi}_1(X_{\overline{k}}; \vec{v}, \vec{w})$, that is the torsor of pro-paths from $\vec{v}$ to $\vec{w}$. In other words we wish to view each natural isomorphism $\gamma : \mathcal{F}_{\vec{v}} \xrightarrow{\sim} \mathcal{F}_{\vec{w}}$ as an element of $\widehat{\pi}_1(X, \vec{v}, \vec{w}) = \mathbf{Iso}(\mathcal{F}_{\vec{v}}, \mathcal{F}_{\vec{w}})$.

To that end let $\pi : Y \to X$ be a finite étale cover with function field F over $\overline{\mathbb{Q}}(t)$, and let $\varphi \in \mathcal{E}_{\vec{v}}$ be an embedding $\varphi : F \hookrightarrow \mathcal{P}_{\vec{v}}(\overline{\mathbb{Q}}(Y)) \hookrightarrow \overline{\mathbb{Q}}\{\{t_{\vec{01}}\}\}$. Let $\gamma \in \widehat{\pi}_1(X^{\mathbf{an}}; \vec{v}, \vec{w})$ be a path from $\vec{v}$ to $\vec{w}$, and let $f \in F$. As before we define the monodromy action of γ on f by analytic continuation

$$\gamma \cdot \varphi(f) = \varphi(\gamma^{-1} \cdot f) \in \mathcal{P}_{\vec{w}}(\overline{\mathbb{Q}}(Y))$$

In other words $\varphi(f) \in \mathcal{P}_{\vec{v}}(\overline{\mathbb{Q}}(Y)) \subseteq \overline{\mathbb{Q}}\{\{t_{\vec{v}}\}\}$ defines a germ of analytic function on the sector $\mathcal{S}_{\vec{v}}$ which can be analytically continued along the path γ to a germ of analytic section $\gamma \cdot \varphi(f) \in \mathcal{P}_{\vec{w}}(\overline{\mathbb{Q}}(Y))$ on the sector $\mathcal{S}_{\vec{w}}$. Moreover the assignment of each embedding $\varphi \in \mathcal{E}_{\vec{v}}(\overline{k}(Y))$ to the deformed embedding $\gamma \cdot \varphi \in \mathcal{E}_{\vec{w}}(\overline{k}(Y))$ is natural with respect to the finite étale cover $\pi : Y \to X$..

Lemma 2.30. *For $\gamma \in \widehat{\pi}_1(X; \vec{v}, \vec{w})$ the assignment of each $\varphi \in \mathcal{E}_{\vec{v}}$ to the analytic continuation $\gamma \cdot \varphi \in \mathcal{E}_{\vec{w}}$ defines a natural isomorphism $\gamma : \mathcal{E}_{\vec{v}} \xrightarrow{\sim} \mathcal{E}_{\vec{w}}$.*

Proof. Let $\pi' : Y' \to Y$ be another finite étale cover with function field $F \hookrightarrow F'$, let $\varphi \in \mathcal{E}_{\vec{v}}(\overline{\mathbb{Q}}(Y))$, and let $\gamma \in \widehat{\pi}_1(X; \vec{v}, \vec{w})$. Let $\mathcal{E}_{\vec{v}}(\pi')(\varphi) = \varphi \circ \pi' \in \mathcal{E}_{\vec{v}}(\overline{\mathbb{Q}}(Z))$. It suffices to show that $\mathcal{E}_{\vec{v}}(\pi')(\gamma \cdot \varphi) = \gamma \cdot \mathcal{E}_{\vec{v}}(\pi')(\gamma)$. But this is immediate: each $e \in \mathcal{F}_{\vec{v}}(Z)$ uniquely determines a lift $\widetilde{\gamma}$ of γ to $\overline{Z}$, in which case $\pi'(\widetilde{\gamma})$ is a lift of γ to $\overline{Y}$ which is natural in π' and γ. The result follows. □

Consequently we can regard the path γ as an element of $\widehat{\pi}_1(X; \vec{v}, \vec{w}) = \mathbf{Iso}(\mathcal{F}_{\vec{v}}, \mathcal{F}_{\vec{w}})$ defined thereby. We can define an action of $\mathbf{Gal}(\overline{k}/k)$ on $\widehat{\pi}_1(X; \vec{v}, \vec{w})$ as follows: let $\varphi \in \mathcal{E}_{\vec{v}}(\overline{k}(Y))$ and $\gamma \in \widehat{\pi}_1(X; \vec{v}, \vec{w})$; then the action of $\sigma \in \mathbf{Gal}(\overline{k}/k)$ is given by the conjugation action $\sigma \cdot \gamma = \sigma\gamma\sigma^{-1}$, that is the element

$$\sigma \cdot \gamma(\varphi) = \sigma \cdot (\gamma \cdot (\sigma^{-1} \cdot \varphi)) \in \mathcal{E}_{\vec{w}}(\overline{k}(Y))$$

In other words for $f \in F$ a test function and $\varphi \in \mathcal{E}_{\vec{v}}(\overline{k}(Y))$ and $\varphi(f) \in \mathcal{P}_{\vec{v}}(\overline{k}(Y))$ we compute $(\sigma \cdot \gamma)(\varphi)(f) = \sigma \cdot (\gamma \cdot (\sigma^{-1} \cdot \varphi))(f)$ by applying σ^{-1} to the coefficients of $\varphi(f)$, analytically continuing along γ, and applying σ to the coefficients of $\gamma \cdot (\sigma^{-1} \cdot \varphi)(f)$. Then we have $(\sigma \cdot \gamma)(\varphi) \in \mathcal{E}_{\vec{w}}(\overline{k}(Y))$ since the actions of γ and σ commute with elements of $\overline{k}(t) \hookrightarrow \mathcal{P}_{\vec{w}}(\overline{k}(t))$ so that we obtain an embedding $\sigma \cdot \gamma(\varphi) : F \hookrightarrow \mathcal{P}_{\vec{w}}(\overline{k}(Y)) \hookrightarrow \overline{k}\{\{t_{\vec{w}}\}\}$. Moreover this is natural with respect to the finite étale cover $\pi : Y \to X$ as required:

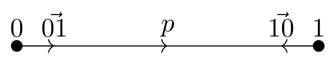
Lemma 2.31. *Let $\gamma \in \widehat{\pi}_1(X; \vec{v}, \vec{w})$ be a pro-path identified with a natural transformation $\gamma : \mathcal{E}_{\vec{v}} \xrightarrow{\sim} \mathcal{E}_{\vec{w}}$. Then for $\sigma \in \mathbf{Gal}(\overline{k}/k)$ the action of σ on γ defines a natural transformation $\sigma \cdot \gamma : \mathcal{E}_{\vec{v}} \xrightarrow{\sim} \mathcal{E}_{\vec{w}}$ and hence an element $\sigma \cdot \gamma \in \widehat{\pi}_1(X; \vec{v}, \vec{w})$.*

Proof. It suffices to show that for $\sigma, \tau \in \mathbf{Gal}(\overline{k}/k)$ and for $\gamma \in \widehat{\pi}_1(X_{\overline{k}}; \vec{v}, \vec{w})$ and $\delta \in \widehat{\pi}_1(X_{\overline{k}}; \vec{u}, \vec{v})$ that $(\sigma \circ \tau) \cdot \gamma = \sigma \cdot (\tau \cdot \gamma)$ and that $\sigma \cdot (\gamma \circ \delta) = (\sigma \cdot \gamma) \circ (\sigma \cdot \delta)$. But this is immediate from the definition. ☐

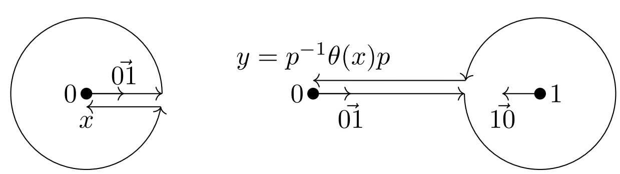
Consequently this yields a Galois representation in each pro-path torsor based at k-rational tangential basepoints $\vec{v}$ and $\vec{w}$ in X

$$\rho_{\vec{v},\vec{w}} : \mathbf{Gal}(\overline{k}/k) \to \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{k}}, \vec{v}, \vec{w}))$$

Finally, we turn to the case of $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ and its base change $X_{\overline{\mathbb{Q}}} = \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ with respect to the tangential basepoints $\mathcal{B} = \{\vec{01}, \vec{10}\}$ associated to the local coordinates $t_{\vec{01}} = t$ and $t_{\vec{10}} = 1 - t$; here $\vec{01}$ and $\vec{10}$ represent the unit tangent vectors $\frac{d}{dt} t_{\vec{01}}|_{t=0}$ and $\frac{d}{dt} t_{\vec{10}}|_{t=0}$ based at 0 and 1 respectively pointing towards 1 and 0 respectively, where t is taken to be an ambient local coordinate for X centered at 0. In this case we have a distinguished straight path $p \in \widehat{\pi}_1(X_{\overline{\mathbb{Q}}}; \vec{01}, \vec{10})$ represented by the standard inverval $(0, 1)$ in $X^{\mathbf{an}}$, and we have an automorphism $\theta(t) = 1 - t$ of X interchanging the local coordinates $t_{\vec{01}} = \theta(t_{\vec{10}})$ and $t_{\vec{10}} = \theta(t_{\vec{01}})$ and hence interchanges the tangential basepoints $\vec{01}$ and $\vec{10}$.



In this case the étale fundamental group $\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{01}) \simeq \widehat{F}_2$ is generated by the loops x and $y = p^{-1}\theta(x)p$ around 0 and 1 respectively, while $\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{10}) \simeq \widehat{F}_2$ is generated by the loops $\theta(x) = pyp^{-1}$ and $\theta(y) = p^{-1}xp$ around 1 and 0 respectively.



Consequently we obtain Galois representations $\rho_{\vec{01}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{01}))$ and $\rho_{\vec{10}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{10}))$ which we can calculate on the distinguished generators $x, y \in \widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{01})$ and $\theta(x), \theta(y) \in \widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{10})$ by the above Puiseux series argument and the $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariance of the isomorphism $\theta : \widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{01}) \xrightarrow{\sim} \widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{10})$. On the other hand, the straight path $p \in \widehat{\pi}_1(X_{\overline{\mathbb{Q}}}; \vec{01}, \vec{10})$ defines an isomorphism $p : \widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{01}) \xrightarrow{\sim} \widehat{\pi}_1(X_{\overline{\mathbb{Q}}}, \vec{10})$ which is not $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant, in which case we obtain a Galois parameter which measures the failure for σ and p to commute with one another:

Definition 2.8. *Let $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and let $\sigma_{\vec{01}} \in \widehat{\pi}_1(X, \vec{01})$ and $\sigma_{\vec{10}} \in \widehat{\pi}_1(X, \vec{10})$ be the respective Galois actions. Then the Galois parameter $\mathfrak{f}_\sigma \in \widehat{\pi}_1(X, \vec{01})$ is the composition*

$$\mathfrak{f}_\sigma = p^{-1}\sigma_{\vec{10}}p\sigma_{\vec{01}}^{-1} \in [\widehat{\pi}_1(X, \vec{01}), \widehat{\pi}_1(X, \vec{01})] \subseteq \widehat{\pi}_1(X, \vec{01})$$

By definition the Galois parameter $\mathfrak{f}_\sigma$ describes the action of σ on the straight path $p \in \widehat{\pi}_1(X; \vec{01}, \vec{10})$ since $\sigma \cdot p = \sigma_{\vec{10}}p\sigma_{\vec{01}}^{-1} = p\mathfrak{f}_\sigma \in \widehat{\pi}_1(X; \vec{01}, \vec{10})$. By abuse of notation it is believable that $\mathfrak{f}_\sigma = [p^{-1}, \sigma]$ should lie in the commutator subgroup of $\widehat{\pi}_1(X, \vec{01})$, although this must be checked more carefully.

Theorem 2.32. *Each $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathbf{Aut}(\widehat{\pi}_1(X, \vec{01})) \simeq \mathbf{Aut}(\widehat{F}_2)$ is determined by parameters $(\chi(\sigma), \mathfrak{f}_\sigma) \in \widehat{\mathbb{Z}}^\times \times [\widehat{F}_2, \widehat{F}_2]$ where $\chi(\sigma) \in \widehat{\mathbb{Z}}^\times$ is the cyclotomic character and $\mathfrak{f}_\sigma \in [\widehat{F}_2, \widehat{F}_2]$ is the non-Abelian Galois symbol in the derived subgroup of $\widehat{F}_2$, acting on the chosen generators $x, y \in \widehat{\pi}_1(X, \vec{01}) \simeq \widehat{F}_2$ by*

$$\begin{cases} \sigma \cdot x = x^{\chi(\sigma)} \\ \sigma \cdot y = \mathfrak{f}_\sigma^{-1} y^{\chi(\sigma)} \mathfrak{f}_\sigma \end{cases}$$

Proof. Consider $f \in \mathcal{M}_{\vec{01}}$ so that $f = \sum_{i \geq -n} a_n t_{\vec{01}}^{i/N} = \sum_{i \geq -n} a_i \exp(\frac{i}{N}\log(t_{\vec{01}}))$ for $a_i \in \overline{\mathbb{Q}}$. Taking the analytic continuation of f along x we obtain

$$x \cdot f = \sum_{i \geq -n} a_i \exp(\tfrac{i}{N}\log(t_{\vec{01}}) + \tfrac{i}{N}2\pi\sqrt{-1}) = \sum_{i \geq -n} a_i \zeta_N^i \exp(\tfrac{i}{N}\log(t_{\vec{01}}))$$

Then $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $x(f)$ by first applying σ^{-1} to f, then by analytic continuation of $\sigma^{-1} \cdot f$ along x, and then by applying σ to $(x\sigma^{-1}) \cdot f$ so that we obtain

$$(\sigma x \sigma^{-1}) \cdot f = \sum_{i \geq -n} a_i \zeta_N^{i\chi(\sigma)} \exp(\tfrac{i}{N}\log(t_{\vec{01}})) = \sum_{i \geq -n} a_i \zeta_N^{i\chi(\sigma)} t_{\vec{01}}^{i/N}$$

By comparing the coefficients of $x \cdot f$ and $(\sigma x \sigma^{-1}) \cdot f$ we obtain $\sigma \cdot x = \sigma x \sigma^{-1} = x^{\chi(\sigma)}$. From this the action of σ on y immediately follows: since $y = p^{-1}\theta(x)p$ and since θ is $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant, we obtain

$$\begin{aligned} \sigma(y) &= \sigma(p)^{-1}\sigma(\theta(x))\sigma(p) = \sigma(p)^{-1}\theta(\sigma(x))\sigma(p) \\ &= \mathfrak{f}_\sigma^{-1}p^{-1}\theta(x^{\chi(\sigma)})p\mathfrak{f}_\sigma = \mathfrak{f}_\sigma^{-1}p^{-1}py^{\chi(\sigma)}p^{-1}p\mathfrak{f}_\sigma = \mathfrak{f}_\sigma^{-1}y^{\chi(\sigma)}\mathfrak{f}_\sigma \end{aligned}$$

It remains to show that $\mathfrak{f}_\sigma \in [\widehat{F}_2, \widehat{F}_2] \subseteq \widehat{F}_2$. To that end it suffices to show that the action of $\mathfrak{f}_\sigma = p^{-1}\sigma_{\vec{10}}p\sigma_{\vec{01}}^{-1} \in \mathbf{Gal}(\Omega/\overline{\mathbb{Q}}(t))$ on $\widehat{\pi}_1(X, \vec{01}) \simeq \widehat{F}_2$ fixes the maximal Abelian quotient $\mathbf{Gal}(\Omega^{\mathbf{ab}}/\overline{\mathbb{Q}}(t)) \simeq \widehat{F}_2^{\mathbf{ab}}$ where $\Omega^{\mathbf{ab}} = \bigcup_{N \geq 1} \overline{\mathbb{Q}}(t_{\vec{01}}^{1/N}, t_{\vec{10}}^{1/N})$ is generated by the cyclic Galois covers over 0 and 1 respectively. On one hand the action of $\mathfrak{f}_\sigma$ on $t_{\vec{01}}^{1/N}$ is given

$$\begin{aligned} t_{\vec{01}}^{1/N} &\xmapsto{\sigma_{\vec{01}}^{-1}} t_{\vec{01}}^{1/N} \overset{p}{\mapsto} (1 - t_{\vec{10}})^{1/N} = \textstyle\sum_{i \geq -n} a_i t_{\vec{10}}^{1/N} && (a_i \in \mathbb{Q}) \\ &\xmapsto{\sigma_{\vec{10}}} \textstyle\sum_{i \geq -n} a_i t_{\vec{10}}^{1/N} = (1 - t_{\vec{10}})^{1/N} \xmapsto{p^{-1}} t_{\vec{01}}^{1/N} \end{aligned}$$

On the other hand the action of $\mathfrak{f}_\sigma$ on $(1-t_{\vec{01}})^{1/N}$ is given

$$(1-t_{\vec{01}})^{1/N} \xmapsto{\sigma_{\vec{01}}^{-1}} (1-t_{\vec{01}})^{1/N} \overset{p}{\mapsto} (1-(1-t_{\vec{10}}))^{1/N} = \textstyle\sum_{i\geq -n} a_i(1-t_{\vec{10}})^{1/N} \qquad (a_i \in \mathbb{Q})$$
$$\xmapsto{\sigma_{\vec{10}}} \textstyle\sum_{i\geq -n} a_i(1-t_{\vec{10}})^{1/N} = (1-(1-t_{\vec{10}}))^{1/N} \xmapsto{p^{-1}} (1-t_{\vec{01}})^{1/N}$$

Since both of these cyclic Galois covers are fixed by the action of $\mathfrak{f}_\sigma$, it follows that $\mathfrak{f}_\sigma$ is contained in the derived subgroup $\widehat{F}_2/\widehat{F}_2^{\mathbf{ab}} = [\widehat{F}_2, \widehat{F}_2]$ as claimed. □

Another way to see the cyclotomic action $\sigma \cdot x = x^{\chi(\sigma)}$ is as follows: consider the place $v : \mathcal{M}_{\vec{01}} \to \overline{\mathbb{Q}} \cup \{\infty\}$ sending $f = \sum_{i\geq -n} a_i t_{\vec{01}}^{i/N}$ to $a_0 \in \overline{\mathbb{Q}}$. Then v is $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant and has the procyclic group $\widehat{\mathbb{Z}}$ generated by x as its inertia group, in which case it follows that $\sigma \cdot x = x^{\chi(\sigma)}$ and that $\sigma \cdot y = \mathfrak{f}_\sigma^{-1} y^{\chi(\sigma)} \mathfrak{f}_\sigma$ as above.

Corollary 2.33. *For $\sigma, \tau \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the parameter $(\chi(\sigma\tau), \mathfrak{f}_{\sigma\tau}) \in \widehat{\mathbb{Z}}^\times \times [\widehat{F}_2, \widehat{F}_2]$ satisfies*

$$(\chi(\sigma\tau), \mathfrak{f}_{\sigma\tau}) = \big(\chi(\sigma)\chi(\tau), \mathfrak{f}_\sigma(x^{\chi(\tau)}, \mathfrak{f}_\tau^{-1} y^{\chi(\tau)} \mathfrak{f}_\tau)\big)$$

Proof. We know that $\chi(\tau \circ \sigma) = \chi(\tau)\chi(\sigma)$ since the cyclotomic character $\chi : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^\times$ is a group homomorphism, and we have

$$\mathfrak{f}_{\sigma\tau} = \sigma \cdot \mathfrak{f}_\tau = \mathfrak{f}_\sigma(x^{\chi(\tau)}, \mathfrak{f}_\tau^{-1} y^{\chi(\tau)} \mathfrak{f}_\tau)$$

since $\mathfrak{f}_{\sigma\tau}(x, y) \in \widehat{F}_2$ is the image of $\mathfrak{f}_\sigma(x, y)$ under the automorphism of $\widehat{F}_2$ sending x to $x^{\chi(\tau)}$ and y to $\mathfrak{f}_\tau^{-1} y^{\chi(\tau)} \mathfrak{f}_\tau$. The result follows. □

The Galois parameter $\mathfrak{f}_\sigma \in [\widehat{F}_2, \widehat{F}_2] \subseteq \widehat{F}_2$ is an unusual object, not least because it is unclear how to write it as an element of $\widehat{F}_2$. On the other hand one can explicitly write $\mathfrak{f}_\sigma$ in each of the finite quotients $\widehat{F}_2/N$ by open normal subgroups N of $\widehat{F}_2$. If we regard the conjugacy class of N as representing a finite étale cover $\pi : Y \to X$ of $X = \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$, then $\mathfrak{f}_\sigma$ completely determines the action of $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the fiber $\pi^{-1}(I)$ for $I = (0, 1)$ the open interval in $\mathbb{P}^1_{\mathbb{C}}$ representing the straight path p. Put another way, the Galois parameter $\mathfrak{f}_\sigma \in [\widehat{F}_2, \widehat{F}_2]$ completely determines the action of $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins, which we turn to below.

3 Dessins D'enfants

When dealing with algebraic curves over $\mathbb{C}$ we have the privilege of being able to draw the associated Riemann surface as a (compact) topological surface, in which case certain geometric features such as the genus can be seen directly. On the other hand, it is less clear how to draw algebraic curves over number fields: certainly we can base change to $\mathbb{C}$ and draw the associated Riemann surface, but in doing so we lose salient geometric data and it is not clear how Galois invariants other than the genus can be seen directly in this way.

The notion of dessins d'enfants gives a partial solution to this problem: by Belyi's theorem an integral proper normal curve X in characteristic 0 admits a finite morphism $\beta : X^{\mathbf{an}} \to \mathbb{P}^1_{\mathbb{C}}$ ramified over $\{0, 1, \infty\}$ precisely if X is defined over a number field, in which case we can produce a bipartite graph $\mathcal{D}$ called the dessin associated to β from which certain Galois invariants of X can be read off directly: for instance the degree of X can be seen as the number of edges in $\mathcal{D}$, and the ramification indices of β can be seen as the degrees of each vertex in $\mathcal{D}$. In fact we will be able to read off the monodromy group and the so called cartography group from such a bipartite graph, which gives a visual way of studying such Galois invariants.

3.1 Belyi's Theorem

Throughout we fix an algebraically closed field k of characteristic 0. Given an integral proper normal curve X over an algebraically closed field k of characteristic 0, recall that by the Riemann correspondence there exists a finite morphism $f : X \to \mathbb{P}^1_k$ which is ramified over finitely many points of $\mathbb{P}^1_k$. We would like to find necessary and sufficient conditions on $f : X \to \mathbb{P}^1_k$ in order for X to be defined over $\overline{\mathbb{Q}}$ or equivalently over some number field. One seemingly obvious characterization due to Weil's rigidity theorem is that $f : X \to \mathbb{P}^1_k$ is defined over $\overline{\mathbb{Q}}$ precisely if its ramification locus is defined over $\overline{\mathbb{Q}}$. We omit the proof and refer the curious reader to [Mumford] and [Wolfart] as this turns out to be rather involved.

Theorem 3.1 (Weil)**.** *Let X be an integral proper normal curve over k. Then X is defined over $\overline{\mathbb{Q}}$ precisely if there exists a finite morphism $f : X \to \mathbb{P}^1_k$ unramified outside $\mathbb{P}^1_{\overline{\mathbb{Q}}}$.*

The key theorem of Belyi reveals a much sharper characterization: given such a finite morphism $f : X \to \mathbb{P}^1_k$ unramified outside $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ we can compose with a sequence of carefully chosen rational functions $f_i : \mathbb{P}^1_k \to \mathbb{P}^1_k$ moving ramification points out to infinity so as to obtain a morphism $\beta : X \to \mathbb{P}^1_k$ unramified outside $\{0, 1, \infty\}$. The functions $f_i : \mathbb{P}^1_k \to \mathbb{P}^1_k$ are chosen so that the composition is unramified outside $\mathbb{P}^1_{\mathbb{Q}}$, and a similar process is applied so that the further composition is unramified outside $\{0, 1, \infty\}$.

Theorem 3.2. *(Belyi) Let X be an integral proper normal curve over an algebraically closed field k of characteristic 0. Then X is defined over $\overline{\mathbb{Q}}$ precisely if there exists a finite morphism $X \to \mathbb{P}^1_k$ unramified outside $\{0, 1, \infty\}$.*

Proof. On one hand if there exists a morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ unramified outside $\{0, 1, \infty\}$ then clearly X is defined over $\overline{\mathbb{Q}}$ by the previous theorem. On the other hand suppose X is defined over $\overline{\mathbb{Q}}$ and let $f_0 : X \to \mathbb{P}^1_{\mathbb{C}}$ be a morphism unramified outside $\mathbb{P}^1_{\overline{\mathbb{Q}}}$. We will first construct a morphism $f : X \to \mathbb{P}^1_{\mathbb{C}}$ unramified outside $\mathbb{P}^1_{\mathbb{Q}}$ as follows: let S be the set of (finite) critical values of f_0 and their $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates, let $f_1(z_1) = \prod_{s \in S}(z_1 - s) \in \mathbb{Q}[z_1]$ and let $f_{i+1}(z_{i+1}) = \mathbf{res}_{z_i}(\frac{df_i}{dz_i}, f_i(z_i) - z_{i+1})$. By construction the roots of f_{i+1} are precisely the critical values of f_i, each f_i is defined over $\mathbb{Q}$, and we have $\mathbf{deg}(f_{i+1}) < \mathbf{deg}(f_i)$ so that there exists some $n > 0$ such that $\mathbf{deg}(f_n) = 0$. Let $f = f_{n-1} \circ ... \circ f_1 \circ f_0$. By construction f is unramified outside $\mathbb{P}^1_{\mathbb{Q}}$ since the set of critical values of $f_i \circ f_{i+1}$ is the union of critical values of f_i and the images of the critical values of f_{i+1} under f_i.

Now let T be the set of critical values of f. If $|T| \leq 3$ we can choose a Möbius transformation g so that $g(T) \subseteq \{0, 1, \infty\}$ in which case $\beta = g \circ f : X \to \mathbb{P}^1_k$ is the desired morphism. So suppose $|T| > 3$. Choose three ordered points of T and choose $n, m \in \mathbb{N}$ and a Möbius transformation sending these three points to 0, $n/(n+m)$, and 1 respectively. Now consider the morphism $g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $g(z) = (n+m)^{n+m} z^m (1-z)^n / n^n m^m$. Then $g(0) = g(1) = 0$ and $g(n/(n+m)) = 1$, in which case $\beta = g \circ f : X \to \mathbb{P}^1_k$ is a finite morphism with strictly fewer critical values than f. We may repeat this process finitely many times to obtain a finite morphism $\beta : X \to \mathbb{P}^1_k$ unramified outside $\{0, 1, \infty\}$ as desired. The result follows. □

Remark 3.1. *It is worth pointing out that Belyi's theorem is unique to characteristic* 0*. Specifically for* k *an algebraically closed field of characteristic* $p > 0$ *every integral proper normal curve* X *over* k *admits a finite morphism* $\beta : X \to \mathbb{P}^1_k$ *unramified outside* ∞*. To see this we note that by the existence of Artin-Schreier covers in positive characteristic there is no obstruction to constructing the appropriate finite morphisms* $\mathbb{P}^1_k \to \mathbb{P}^1_k$ *which we used in the above proof to push the ramification to* ∞*. On the other hand, there is an analog of Belyi's theorem in characteristic* $p > 2$ *using the tame fundamental group: an integral proper normal curve* X *over* k *admits a tamely ramified finite morphism* $\beta : X \to \mathbb{P}^1_k$ *unramified outside* $\{0, 1, \infty\}$ *precisely if* X *is defined over* $\overline{\mathbb{F}}_p$*. The case of characteristic* $p = 2$ *is open.*

We say that a rational function $\beta : X \to \mathbb{P}^1_k$ is a Belyi function if its critical values are contained in $\{0, 1, \infty\}$. We say that two Belyi functions $\beta : X \to \mathbb{P}^1_k$ and $\gamma : Y \to \mathbb{P}^1_k$ are isomorphic if there exists an isomorphism $\varphi : X \xrightarrow{\sim} Y$ such that $\beta = \varphi \circ \gamma$. By the usual covering theory, such a rational function determines a finite étale morphism $\beta : X \to \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ realizing X as an unramified cover, and conversely every finite étale morphism $\beta : X \to \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ extends uniquely to a finite morphism $\beta : X \to \mathbb{P}^1_k$ realizing X as a branched cover ramified over $\{0, 1, \infty\}$. Moreover such an unramified cover is determined by an open subgroup G of the étale fundamental group $\widehat{\pi}_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\}) \simeq \widehat{F}_2$, or equivalently a subgroup of finite index in the free group $F_2 = \langle x, y, z \mid xyz = 1\rangle$, in which case the ramification indices over $\{0, 1, \infty\}$ are determined by the lengths of orbits in $\widehat{\pi}_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})/G$ under the canonical action of x, y, and z respectively. From this we obtain the following equivalent characterization:

Corollary 3.3. *There is a bijection between conjugacy classes of open subgroups of the étale fundamental group* $\widehat{\pi}_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\}) \simeq \widehat{F}_2$*, or rather conjugacy classes of subgroups of finite index in the fundamental group* $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}) \simeq F_2$*, and isomorphism classes of Belyi functions* $\beta : X \to \mathbb{P}^1_k$*.*

We can also refine this situation by bounding the ramification indices above the points $\{0, 1, \infty\}$. The most relevant restriction we can make is to bound the ramification order above 1 to be at most 2, or even exactly 2, the importance of which will later become clear in the context of Grothendieck's theorem on dessins.

Theorem 3.4. *Let* X *be an integral proper normal curve over an algebraically closed field* k *of characteristic* 0*. Then* X *is defined over* $\overline{\mathbb{Q}}$ *precisely if there exists a morphism* $X \to \mathbb{P}^1_k$ *étale outside* $\{0, 1, \infty\}$ *such that all points over* $1 \in \mathbb{P}^1_k$ *have ramification order precisely* 2*.*

Proof. Let $\beta : X \to \mathbb{P}^1_k$ be a finite morphism unramified outside $\{0, 1, \infty\}$ which exists precisely if X is defined over $\overline{\mathbb{Q}}$. Then the finite morphism $4\beta(1 - \beta) : X \to \mathbb{P}^1_k$ is étale outside $\{0, 1, \infty\}$ with the prescribed ramification. □

We say that a Belyi function $\beta : X \to \mathbb{P}^1_k$ is pre-clean if its ramification orders above 1 are each at most 2, and clean if its ramification orders above 1 are each exactly 2. Again by the usual covering theory we obtain the following equivalent characterization of (pre)-clean Belyi functions:

Corollary 3.5. *There is a bijection between the set of conjugacy classes of open subgroups of the étale fundamental group* $\widehat{\pi}_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\})/(y^2) \simeq \widehat{F}_2/\langle y^2\rangle$*, or rather conjugacy classes of subgroups of finite index in the fundamental group* $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\})/(y^2) \simeq F_2$*, and the set of isomorphism classes of pre-clean Belyi morphisms.*

We will give only one example of a Belyi morphism for now, although it is an important one: given an elliptic curve defined over a number field, one can produce a Belyi function so that the associated complex elliptic curve is realized as a branched cover of $\mathbb{P}^1_{\mathbb{C}}$ ramified outside $\{0, 1, \infty\}$. Since such an elliptic curve is completely determined by its j-invariant, we will obtain a Galois action on such Belyi morphisms induced by the usual Galois action.

Example 3.1. *Let X be an elliptic curve given for $\lambda \in \mathbb{C} \setminus \{0, 1\}$ in Legendre form as the zero locus of $y^2 - x(x-1)(x-\lambda)$ in $\mathbb{A}^2_{\mathbb{C}}$ and extended projectively. Then X is defined over $\overline{\mathbb{Q}}$ precisely if $\lambda \in \overline{\mathbb{Q}}$, precisely if its j-invariant $j = 256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} \in \overline{\mathbb{Q}}$, in which case we can define a Belyi function $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ by composing the coordinate projection $\pi(x, y) = x$ with some rational function $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. Likewise let X be an elliptic given for $p, q \in \mathbb{C} \setminus \{0\}$ in Weierstrass form as the zero locus of $y^2 - x^3 - px - q$ in $\mathbb{A}^2_{\mathbb{C}}$ and extended projectively. Then X is defined over $\overline{\mathbb{Q}}$ precisely if its j-invariant $j = 1728\frac{4p^3}{4p^3+27q^2} \in \overline{\mathbb{Q}}$, in which case there exists a Belyi function $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ as above.*

For instance, let X be given by $\lambda = 1 + \sqrt{2}$ in Legendre form. Then λ is algebraic with minimal polynomial $f(x) = x^2 - 2x - 1$ over $\mathbb{Q}$, and the function $-f(x)$ sends the critical values of $\pi(x, y) = x$ to 1, 2, ∞, and 0 respectively; consequently the map $g(x) = x^2/4(x-1)$ sends these to ∞, 1, ∞, and 0 respectively, whence the function

$$\beta(x, y) = \frac{(-f(x))^2}{4(-f(x)-1)} = \frac{(x^2-2x-1)^2}{4x(2-x)}$$

By construction this function is unramified outside $\{0, 1, \infty\}$, hence it is a Belyi function of degree 8. Moreover this is defined over $\mathbb{Q}(\sqrt{2})$, and so the Galois action sends the above elliptic curve with $\lambda = 1 + \sqrt{2}$ to the conjugate elliptic curve with $\lambda' = 1 - \sqrt{2}$.

By the uniformization theorem we obtain another equivalent statement of Belyi's theorem which characterize those compact Riemann surfaces defined over $\overline{\mathbb{Q}}$ in terms of certain Fuchsian groups. Recall that for G a subgroup of finite index in $\mathbf{PSL}_2(\mathbb{R})$, we obtain an open Riemann surface $\mathbb{H}//G$ which we can compactify by adding finitely many cusps corresponding to the orbits of G on the rational projective line $\mathbb{P}^1_{\mathbb{Q}} \subseteq \mathbb{P}^1_{\mathbb{C}}$ so as to obtain a compact Riemann surface $\overline{\mathbb{H}//G}$.

Theorem 3.6. *Let X be a compact Riemann surface. Then X is defined over $\overline{\mathbb{Q}}$ precisely if $X \simeq \overline{\mathbb{H}//G}$ for some subgroup G of finite index in a triangle group Δ in $\mathbf{PSL}_2(\mathbb{R})$.*

In this case we obtain a Belyi function $\beta : X \simeq \overline{\mathbb{H}//G} \to \overline{\mathbb{H}//\Delta} \simeq \mathbb{P}^1_{\mathbb{C}}$ induced by the Δ-automorphic Schwartz triangle function $J_\Delta : \mathbb{H} \to \mathbb{P}^1_{\mathbb{C}}$, with the degree of β is equal to the index of G in Δ. We have a similar result corresponding to (pre)clean Belyi pairs:

Corollary 3.7. *Let X be a compact Riemann surface. Then X is defined over $\overline{\mathbb{Q}}$ precisely if $X \simeq \overline{\mathbb{H}//G}$ for some subgroup G of finite index in Γ, precisely if $X \simeq \overline{\mathbb{H}//G}$ for some subgroup G of finite index in $\Gamma(2)$.*

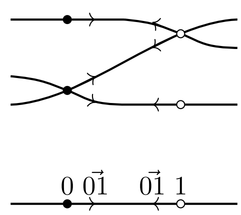
In this case we obtain Belyi functions $\beta : X \simeq \overline{\mathbb{H}//G} \to \overline{\mathbb{H}//\Gamma} \simeq \mathbb{P}^1_{\mathbb{C}}$ and $\beta : X \simeq \overline{\mathbb{H}//G} \to \overline{\mathbb{H}//\Gamma(2)} \simeq \mathbb{P}^1_{\mathbb{C}}$ induced by the modular J-function $J : \mathbb{H} \to \mathbb{P}^1_{\mathbb{C}}$ and the modular λ-function $\lambda : \mathbb{H} \to \mathbb{P}^1_{\mathbb{C}}$ respectively; as before the degree is equal to the index of G. Just as there is a canonical $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Belyi pairs coming from the Galois action on curves over number fields, so too is there a canonical $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on such (arithmetic) subgroups of finite index in Γ and $\Gamma(2)$, and indeed these Galois actions coincide.

3.2 Dessins D'enfants

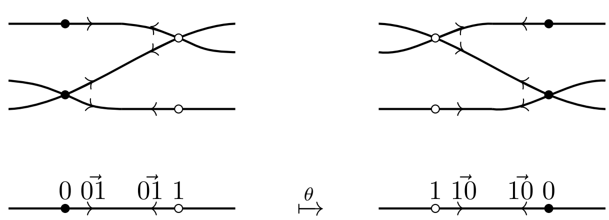
A good way of visualizing a Belyi pair (X, β) and hence a good way of seeing how $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on it, is to take some simple combinatorial structure on $\mathbb{P}^1_{\mathbb{C}}$, say a triangulation or a (hyper)map, and to lift it to X via the Belyi function $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$. Then the action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Belyi pairs lifts to an action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on these combinatorial structures. The goal, albeit a challenging one, is to characterize $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in terms of its action on these combinatorial objects.

Consider the standard triangulation $\mathcal{T}_0$ of $\mathbb{P}^1_{\mathbb{C}}$, that is the triangulation by vertices $\{0, 1, \infty\}$, by edges represented by the standard intervals $(0, 1)$, $(1, \infty)$, and $(-\infty, 0)$, and by faces represented by the upper and lower half planes. If $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is a Belyi function (of degree N then this triangulation lifts to a triangulation $\mathcal{T}$ of X; since β is unramified outside of $\{0, 1, \infty\}$ the two faces and three edges of this triangulation lift to $2N$ faces and $3N$ edges in $\mathcal{T}$. Moreover if the N sheets over $v \in \{0, 1, \infty\}$ join in cycles of length $n_{v,1}, ..., n_{v,k_v}$ so that $n_{v,1}, ..., n_{v,k_v} = N$, we have that $\beta^{-1}(v)$ consists of k_v vertices with valency $2n_{v,1}, ..., 2n_{v,k_v}$. Indeed this yields the Euler characteristic $2g - 2 = k_0 + k_1 + k_\infty - N$ as expected.

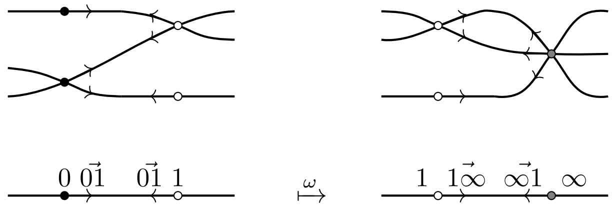
Suppose now that we only lift part of the triangulation $\mathcal{T}_0$ to X, say the vertices $\{0, 1\}$ and the interval $(0, 1)$. Then by taking the preimage $\beta^{-1}[0, 1]$ in X we obtain a bipartite graph whose vertices are the preimages of the endpoints 0 and 1 colored black and white respectively, and whose edges are the preimages of the interval $(0, 1)$. Conversely if we specify the ramification above 0 and 1, or equivalently the fiber over $\vec{01}$ and $\vec{10}$, then such a bipartite graph is equivalent to a choice of lifts of the straight path p connected each edge of type $\vec{01}$ to an edge of type $\vec{10}$.



Of course we can also obtain a bipartite graph by taking the preimages $\beta^{-1}[1, \infty]$ and $\beta^{-1}[\infty, 0]$, and we could have obtained a different bipartite structure by switching the coloring of the vertices. Alternatively, we can choose only to consider the preimage $\beta^{-1}[0, 1]$ from which we can recover the other preimages by pulling back along certain automorphisms of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$. For instance consider the automorphism $\theta(t) = 1 - t$ of $\mathbb{P}^1_{\mathbb{C}}$ involuting 0 and 1; pulling back β along θ and taking the preimage $(\theta^* \beta)^{-1}[0, 1]$ is equivalent to the preimage $\beta^{-1}[0, 1]$ but with the colors reversed.



Likewise consider the automorphism $\omega(t) = \frac{1}{1-t}$ of $\mathbb{P}^1_{\mathbb{C}}$ cyclically permuting 0, 1, and ∞; pulling back β along ω and taking the preimage $(\omega^* \beta)^{-1}[0, 1]$ is equivalent to the preimage $\beta^{-1}[1, \infty]$, and similarly pulling back β along ω^2 and taking the preimage $((\omega^2)^* \beta)^{-1}[0, 1]$ is equivalent to the preimage $\beta^{-1}[\infty, 0]$.



Indeed since $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is a Belyi function and since $\theta, \omega : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ preserve the critical points of β under postcomposition, the pullbacks $\theta^*\beta, \omega^*\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ are Belyi functions. For this reason we can justify defining the dessin associated to a Belyi function $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ in terms of the lift $\beta^{-1}[0,1]$, and the other lifts can be recovered accordingly.

Definition 3.1. *Let $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ be a Belyi function. Then the dessin $\mathcal{D}$ associated to β is the preimage $\beta^{-1}[0,1]$ regarded as a bipartite graph.*

Now if $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is a pre-clean Belyi function this definition simplifies substantially. In this case we can regard $\beta^{-1}[0,1]$ as a graph whose vertices are the preimage $\beta^{-1}(0)$ and whose edges are in bijection with the preimage $\beta^{-1}(1)$ in the sense that each $y \in \beta^{-1}(1)$ has at most 2 edges of type $\vec{10}$ which define an edge with y as a midpoint. Indeed given such a graph we can obtain a unique bipartite graph by placing a black vertex at each vertex and placing a white vertex at the midpoint of each edge.

Definition 3.2. *Let $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ be a pre-clean Belyi function. Then the dessin $\mathcal{D}$ associated to $\mathcal{D}$ is the preimage $\beta^{-1}[0,1]$ regarded as a graph in the above sense.*

Indeed since every Belyi function $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ yields a clean dessin $\widetilde{\beta} : X \to \mathbb{P}^1_{\mathbb{C}}$ with $\widetilde{\beta} = 4\beta(1-\beta)$, every dessin $\mathcal{D}$ yields a clean dessin $\widetilde{\mathcal{D}}$ by the above midpoint procedure. Given a dessin $\mathcal{D}$ represented by a Belyi function $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$, we say that $\theta \circ \beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is the recolored dessin with the black and white vertices of $\mathcal{D}$ exchanged, and we say that $\omega^{-1} \circ \beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is the dual dessin with the white vertices replaced with black vertices and with white vertices given by the vertices at infinity at the center of each face.

One of the convenient features of dessins is that the monodromy group of the associated covering space of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ can be read off directly from the dessin. Specifically, let $\mathcal{D}$ be a dessin of degree n with a choice of labeling, that is a choice of bijection between the edges of $\mathcal{D}$ and the set $\{1, ..., n\}$. Since the dessin is implicitly embedded in a Riemann surface we obtain a cyclic ordering of edges around each vertex of the dessin. Now let $\sigma_0 \in S_n$ be the permutation given by the product of disjoint cycles of edges around each black vertex, and let $\sigma_1 \in S_n$ be the permutation given by the product of disjoint cycles of edges around each white vertex, all with the same orientation. Then these permutations generate the monodromy group of $\mathcal{D}$:

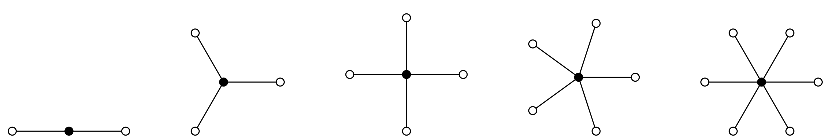
Definition 3.3. *Let $\mathcal{D}$ be a dessin of degree n. The monodromy group of $\mathcal{D}$ is the subgroup of S_n generated by the permutations σ_0 and σ_1 of the edges of $\mathcal{D}$.*

Indeed since we can associate to each dessin $\mathcal{D}$ a clean dessin $\widetilde{\mathcal{D}}$ by the above midpoint procedure, we can associate to $\widetilde{\mathcal{D}}$ its monodromy group which we call the cartography group of $\mathcal{D}$. The cartography group of a dessin is typically a much stronger invariant than the monodromy group, at the cost of being much more difficult to compute: whereas the monodromy group consists of at most $n!$ elements, the cartography group consists of at most $(2n)!$ elements.

Definition 3.4. *Let $\mathcal{D}$ be a dessin of order n, and let $\widetilde{\mathcal{D}}$ be the clean dessin of order $2n$ associated to $\mathcal{D}$. The cartography group of $\mathcal{D}$ is the subgroup of S_{2n} given by the monodromy group of the associated clean dessin $\widetilde{\mathcal{D}}$.*

At this point we are overdue for examples. We will focus on some elementary examples, namely those of the cyclic dessins, the Chebyshev dessins, the Fermat dessins, and the dessins associated to modular curves, which appear in [Jones-Streit] and [Shabat-Voevodsky]. The curious reader is encouraged to consult these for more examples, as well as [Swinerton-Dyer] for a comprehensive list of small dessins and their fields of definition.

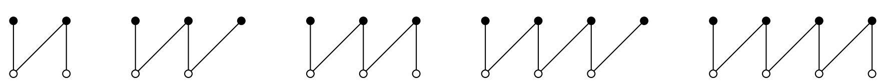
Example 3.2. *Let $X = \mathbb{P}^1_{\mathbb{C}}$ and let $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ be given by $\beta(z) = z^n$; this is the standard cyclic Galois cover of $\mathbb{P}^1_{\mathbb{C}}$ which is unramified outside $\{0, \infty\}$, hence it is a Belyi function of degree n. By change of variables the functions $\beta \circ \theta : X \to \mathbb{P}^1_{\mathbb{C}}$ and $\beta \circ \omega : X \to \mathbb{P}^1_{\mathbb{C}}$ given by $(\beta \circ \theta)(z) = (1 - z)^n$ and $(\beta \circ \omega)(z) = (1 - z)^{-n}$ are Belyi functions of degree n. Indeed all the above Belyi functions are defined over $\mathbb{Q}$, and hence are invariant under the canonical $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. By construction β has ramification order n over 0 and is unramified over 1, which completely determines the associated dessin: this has one vertex of type 0 with order n and n vertices of type 1 each with order 1, hence we have n edges of type $\vec{01}$ with the obvious incidence relations. For instance we can draw the associated dessin for $n = 2, 3, 4, 5, 6$ and indeed these are planar.*



It is easy to see that the monodromy group of the cyclic dessin of order n is the cyclic group $\mathbb{Z}/n\mathbb{Z}$: for the above dessins the monodromy group G is generated by the single cyclic permutation of n edges around the single black vertex. The cartography group is more complicated but still readily computed: in this case the cartography group C is a transitive subgroup of S_{2n} generated by a single n-cycle and by a product of n disjoint transpositions; then we have two disjoint n-cycles r_0 and r_1 which generate an index 2 subgroup $\langle r_0 \rangle \times \langle r_1 \rangle \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ of C on which the transposition acts by permuting the two factors, in which case it follows that C is isomorphic to the wreath product $\mathbb{Z}/n\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$ of order $2n^2$.

Example 3.3. *Let $T_n(x)$ be the n-th Chebyshev polynomial, determined by the recurrence relations $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. Then $T_n(x)$ has critical values contained in $\{\pm 1\}$ so T_n is a Shabat polynomial, and hence the functions $(T_n(x)+1)/2$ and $T_n^2(x)$, and $1 - T_n^2(x)$ are Belyi polynomials of degrees n, $2n$, and $2n$ respectively. Since $T_n(x)$ has only simple roots $1 - T_n^2(x)$ is a clean Belyi polynomial. By the above recurrence relation $T_n(x)$ is defined over $\mathbb{Q}$, and hence is invariant under the canonical $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, as are the above Belyi polynomials.*

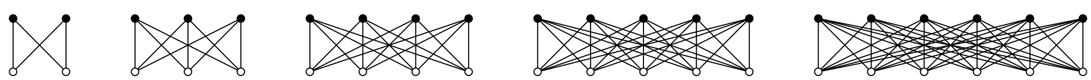
For instance we can draw the dessins associated to T_n for $n = 4, 5, 6, 7, 8$:



Indeed the even Chebyshev dessins are invariant under recoloring, whereas the odd Chebyshev dessins are not, as they have more black vertices than white vertices. It is easy to compute the monodromy group G in this case: G must contain the cyclic subgroup $\langle \sigma_\infty \rangle \simeq \mathbb{Z}/n\mathbb{Z}$ which has index 2 in G and hence is normal; then one of σ_0 or σ_1 must have a fixed point, which we may assume to be σ_0 up *to recoloring, in which case we can label the edges left to right by elements of $\mathbb{Z}/n\mathbb{Z}$ so that σ_0 acts by translation and σ_0 fixes the edge labeled 0. Then the stabilizer of this edge has index n in G and hence σ_0 must be an involution; but then σ_0 normalizes $\langle\sigma_\infty\rangle \simeq \mathbb{Z}/n\mathbb{Z}$ acting by multiplication by -1 for $n \geq 3$, in which case it follows that $G = D_n$ is the dihedral group of order $2n$. Consequently the cartography group $C = D_{2n}$ is the dihedral group of order $4n$, since the associated pre-clean dessin is simply the Chebyshev dessin with twice the number of vertices.*

Example 3.4. *Let X be the Fermat curve of degree n given as the zero locus of $x^n + y^n = 1$ in $\mathbb{A}^2_{\mathbb{C}}$ and extended projectively. By the genus degree formula this is a compact Riemann surface of genus $(n-1)(n-2)/2$ defined over $\mathbb{Q}$. Although the coordinate projection $\beta(x, y) = x$ does not define a Belyi function since its critical values are the n-th roots of unity, the function $\beta(x, y) = x^n$ is unramified outside $\{0, 1, \infty\}$, hence it is a Belyi function of degree n^2.*

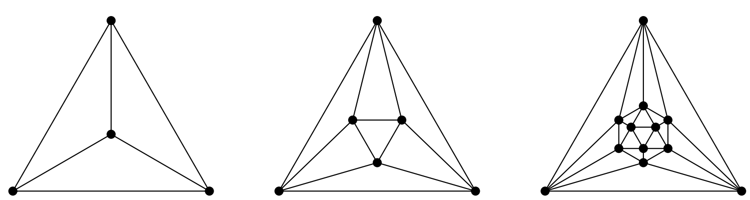
It is easy to see that this ramification data completely fixes the dessin associated to the Fermat curve: since n^2 edges must uniquely connect n edges to n other edges, the resulting dessin is the complete bipartite graph of order $2n$. For instance we can draw the associated dessin for $n = 2, 3, 4, 5, 6$ of genus $g = 0, 1, 3, 6, 10$:



Indeed the Fermat dessins are invariant under recoloring since complete bipartite graphs are symmetric in their vertices, and are self-dual since the lift of the unviersal triangulation $\mathcal{T}_0$ of $\mathbb{P}^1_{\mathbb{C}}$ to the Fermat curve is the complete tripartite graph of order $3n$.

It is easy to see that the monodromy group of the Fermat dessin of order 4 is the Klein 4 group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$: indeed every element is order 2 and generated by the permutations $(12)(34)$ and $(13)(24)$. In general the monodromy group of the Fermat dessin of order n^2 is the cyclic group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ which is easily verified using GAP. The cartography group is harder to compute.

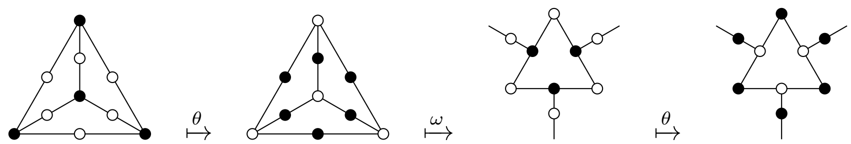
Example 3.5. *Consider the dessins associated to the tetrahedron, the octahedron, and the icosahedron embedded in the Riemann sphere, drawn as follows:*



Indeed we can place white vertices at the midpoint of each edge, and obtain a clean dessin in each case. These have ramification types $(3^4; 2^6)$, $(4^6; 2^{12})$, and $(5^{12}; 2^{30})$, respectively, and indeed using the genus 0 algorithm we obtain Belyi functions in each case. One can use GAP to verify that the tetrahedron dessin has cartography group A_4, the octahedron dessin has cartography group S_4, and the icosahedron dessin has cartography group A_5. For instance the cartography group of the tetrahedron dessin is generated by the permutations $(1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)$ and $(1, 6)(2, 11)(3, 7)(4, 9)(5, 12)(8, 10)$; the other examples are similar, but we leave this as a (tedious) exercise.

Still the most conceptually satisfying descriptions of the above dessins is related to modular curves. Recall that for $\Gamma(n) = \mathbf{ker}(\mathbf{PSL}_2(\mathbb{Z}) \to \mathbf{PSL}_2(\mathbb{Z}/n\mathbb{Z}))$ the principal congruence subgroup of level $n \geq 2$, the affine modular curve is the quotient $Y_n = \mathbb{H}//\Gamma(n)$; Then the canonical projection $Y_n = \mathbb{H}//\Gamma(n) \to \mathbb{H}//\Gamma(1) = Y_1 \simeq \mathbb{P}^1_{\mathbb{C}}$ is a Belyi function. In particular the dessins associated to the modular *curves Y_3, Y_4, and Y_5 correspond precisely to the tetrahedron, the octahedron, and the icosahedron dessins as above, see [Shabat-Voevodsky].*

It is also instructive to see how the standard automorphisms θ and ω of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ act on these dessins. For instance θ and ω act on the tetrahedron dessin as follows:



Indeed the tetrahedron dessin is self dual, but only up to recoloring: we have $(\omega \circ \theta)(\mathcal{D}) = \mathcal{D}$ so that $\omega^{-1}(\mathcal{D}) = (\theta \circ \omega \circ \theta)(\mathcal{D}) = \theta(\mathcal{D})$. Similarly the octahedron dessin has the cube dessin as its dual, while the icosahedron dessin has the dodecahedron dessin as its dual, which we can see by applying the automorphism $\omega^{-1} = \theta \circ \omega \circ \theta$; again we leave this as a (tedious) exercise.

All of the above examples are defined over $\mathbb{Q}$, with the exception of the earlier example of the elliptic curve in which case the dessin can be defined over an arbitrary number field. Still, even the combinatorial theory of dessins defined over $\mathbb{Q}$ is interesting: for instance many of the finite simple groups can be realized as monodromy groups or cartography groups of dessins defined over $\mathbb{Q}$, in which case one can obtain visualizations for objects on which these groups act.

3.3 Galois Orbits of Dessins

From the above considerations it is clear that there is a canonical action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins: for instance if we realize dessins as finite extensions of $\overline{\mathbb{Q}}(t)$ then the canonical action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on coefficients of $\overline{\mathbb{Q}}(t)$ extends to an action on dessins; likewise if we realize dessins as conjugacy classes of subgroups of finite index in the modular group $\Gamma = \mathbf{PSL}_2(\mathbb{Z})$ or the principal congruence subgroup $\Gamma(2)$, then the canonical action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on arithmetic subgroups extends to an action on dessins.

The first and most obvious Galois invariant of a dessin $\mathcal{D}$ is its ramification indices and in particular its degree. Namely for $\mathcal{D}$ a dessin represented by a Belyi function $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ of degree N we have positive integers $k_{0,1}, ..., k_{0,n_0}, k_{1,1}, ..., k_{1,n_1}, k_{\infty,1}, ..., k_{\infty,n_\infty}$ given by the ramification orders at the points above 0, 1, and ∞, such that

$$k_{0,1} + ... + k_{0,n_0} = k_{1,1} + ... + k_{1,n_1} = k_{\infty,1} + ... + k_{\infty,n_\infty} = N$$

By change of coordinates we will restrict to the ramification points above 0, in which case we write $k_1 + ... + k_n = N$ for simplicity. The main theorem is that the action of $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ preserves this partition, and hence preserves the ramification indices.

Theorem 3.8. *Let $\mathcal{D}$ be a dessin and let $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then $\sigma \cdot \mathcal{D}$ has ramification indices equivalent to that of $\mathcal{D}$.*

Proof. Let $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ be a Belyi morphism of degree N with ramification indices $k_1 + ... + k_n = N$ above 0 representing $\mathcal{D}$. Consider the field $\mathcal{M}(\mathbb{P}^1_{\mathbb{C}})$ of meromorphic functions over $\mathbb{P}^1_{\mathbb{C}}$, and consider the valuation subring $\mathcal{M}_0(\mathbb{P}^1_{\mathbb{C}})$ on those meromorphic functions $f \in \mathcal{M}(\mathbb{P}^1_{\mathbb{C}})$ with no pole at 0. Now consider the valuation ideal I_0 of $\mathcal{M}_0(\mathbb{P}^1_{\mathbb{C}})$ consisting of those meromorphic functions $f \in \mathcal{M}(\mathbb{P}^1_{\mathbb{C}})$ vanishing at 0, that is the kernel of the evaluation morphism $\mathbf{ev}_0 : \mathcal{M}_0(\mathbb{P}^1_{\mathbb{C}}) \to \mathbb{C}$ at 0. By definition the Belyi morphism β induces an embedding $\mathcal{M}(\mathbb{P}^1_{\mathbb{C}}) \hookrightarrow \mathcal{M}(X)$ by pre-composition with β; under this embedding the valuation ideal I_0 decomposes as a product $I_0 = \prod_{i=1}^n I_i^{n_i}$ where each I_i is the valuation ideal of a ramification point above 0. But then since $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$ the valuation ideal I_0 descends to the valuation ideal J_0 of $\mathcal{M}_0(\mathbb{P}^1_{\overline{\mathbb{Q}}})$ which decomposes as a product $J_0 = \prod_{i=1}^n J_i^{n_i}$ where each J_i is the valuation ideal of a $\overline{\mathbb{Q}}$-rational ramification point above 0. But the canonical action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathcal{M}(\mathbb{P}^1_{\overline{\mathbb{Q}}})$ preserves ramification, and hence preserves the above product decomposition, whence the claim. □

Corollary 3.9. *Let $\mathcal{D}$ be a dessin of genus g, and let $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ Then $\sigma \cdot \mathcal{D}$ is a dessin of genus g hence the genus is a Galois invariant.*

Proof. This is immediate since the ramification indices k_0, k_1, and k_∞ of $\mathcal{D}$ and the degree N of $\mathcal{D}$ are invariant under σ, and by Riemann-Hurewitz the genus of a Belyi morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ representing $\mathcal{D}$ can be expressed in terms of the ramification indices k_0, k_1, and k_∞ and the degree N, whence the claim. □

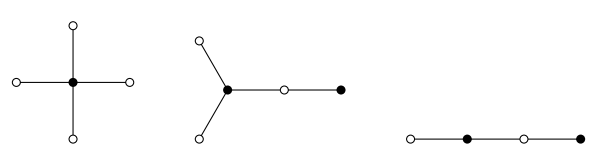
In nice cases this is already enough to separate Galois orbits of dessins. Namely two dessins with different ramification type cannot lie in the same Galois orbit, and in particular if a ramification type is unique among such bipartite graphs then the associated dessin dessin constitutes its own Galois orbit and is defined over $\mathbb{Q}$.

A much stronger invariant is the monodromy group and the cartography group of a dessin; we will not prove this, although the argument is very similar to the above proof albeit much more involved. The complete proof can be found in [Matzat], while a simplified sketch, still rather involved, can be found in [Jones-Streit].

Theorem 3.10. *Let $\beta : X \to \mathbb{P}^1_k$ be a Belyi morphism with monodromy group G and cartography group C, and let $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then the monodromy group $\sigma \cdot G$ and the cartography group $\sigma \cdot C$ of the Belyi morphism $\sigma \cdot \beta : X \to \mathbb{P}^1_k$ are conjugate to G and C respectively, hence the monodromy group and the cartography group are Galois invariants.*

One typical strategy for separating dessins of the same ramification type into their Galois orbits is to show that their monodromy groups or cartographic groups differ. Namely, given a fixed ramification type we can list all the bipartite graphs with this ramification type and compute their monodromy and cartography groups; any such bipartite graphs with different monodromy or cartography groups cannot lie in the same Galois orbit, and in particular if a monodromy or cartography group is unique among such bipartite graphs then the associated dessin constitutes its own Galois orbit and is defined over $\mathbb{Q}$.

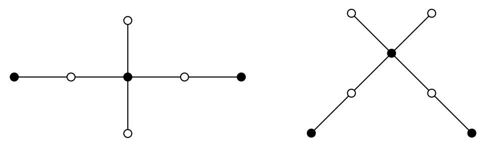
Example 3.6. *Let $\mathcal{D}$ be a plane tree dessin of degree 4. There are three such dessins, with ramification types $(4;1,1,1,1)$, $(3,1;2,1,1)$, and $(2,2;2,1,1)$ respectively, and in each case the monodromy group G is a transitive subgroup of S_4 acting on edges.*



The first of these has monodromy group $\mathbb{Z}/4$ and the last of these has monodromy group D_4 as we have seen previously. It remains to exhibit the monodromy group of the second of these: here σ_0 acts as a

3-cycle and σ_1 acts as a 2-cycle which generate a group of order $|G| = |S_4| = 24$, hence $G = S_4$. Indeed since each of the three trees have different monodromy groups it follows that the associated dessins are contained in separate Galois orbits and hence are defined over $\mathbb{Q}$.

Example 3.7 (Jones-Striet). *Let $\mathcal{D}$ be a plane tree dessin of degree 6 with ramification type $(4,1,1;2,2,1,1)$, so that the monodromy group G is a transitive subgroup of S_6 acting on the edges. One can easily see that there are only two such plane trees, namely:*



In the first case we can use the rotational symmetry of the plane tree to partition the six edges into three pairs so that σ_0 and σ_1 act on these pairs; then G is imprimitive and is contained in the wreath product $S_2 \wr S_3$, that is the largest subgroup of S_6 on those permutations preserving these pairs. But then σ_0 and σ_1 generate a group of order $|G| = |S_2 \wr S_3| = 48$, hence $G = S_2 \wr S_3$. Here the rotational symmetry corresponds to the central involution of $S_2 \wr S_3$ under which the above tree is invariant.

In the second case we can label the edges by points in $\mathbb{P}^1_{\mathbb{F}_5}$ so that σ_0 and σ_1 act as $\left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$ respectively. But then σ_0 and σ_1 generate a group of order $|G| = |\mathbf{PGL}_2(\mathbb{F}_5)| = 120$, hence $G = \mathbf{PGL}_2(\mathbb{F}_5)$. Here the reflection symmetry corresponds to the pair (σ_0, σ_1) being conjugate to the pair $(\sigma_0^{-1}, \sigma_1^{-1}) = (\sigma_0^{-1}, \sigma_1)$ in S_6, under which the above tree is invariant. Indeed since both trees have different monodromy groups it follows that the associated dessins are contained in separate Galois orbits and hence are defined over $\mathbb{Q}$.

Remark 3.2. *Of course neither the monodromy group nor the cartographic group of a dessin is a complete Galois invariants: there are specific examples of plane tree dessins with conjugate monodromy and cartographic groups which nevertheless are contained in separate Galois orbits. Perhaps the most famous examples of this are the so called Leila flowers: these are the two plane tree dessins with ramification type $(5, 1^{22}; 6, 6, 5, 5, 5)$ which are defined over $\mathbb{Q}$ and hence lie in separate Galois orbits, but have conjugate monodromy and cartography groups; see [Schneps] and [Jones-Streit].*

The other basic question one can ask of the Galois action on dessins is the extent to which the associated Galois representation is faithful. This turns out to be the case, as we will show later that both the outer Galois representation $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F}_2)$ as well as the tangential Galois representations $\rho_{\vec{v}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{F}_2)$ are faithful. Before we show this in full generality, it is instructive to see this for lower genera. The genus 1 case is easiest to see since it reduces to the existence of elliptic curves over number fields with prescribed j-invariants.

Theorem 3.11. *The action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of isomorphism classes of genus 1 dessins is faithful.*

Proof. Recall that a genus 1 dessin corresponds to a $\overline{\mathbb{Q}}$-isomorphism class of elliptic curves, which are classified by their j-invariant; such a curve is defined over $\overline{\mathbb{Q}}$ precisely if its j-invariant is contained in $\overline{\mathbb{Q}}$. Now for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ there exists some $j \in \overline{\mathbb{Q}}$ such that σ does not act trivially on j. Let E be an elliptic curve over $\overline{\mathbb{Q}}$ with $j(E) = j$. Then by Belyi's theorem there exists a function $\beta : E \to \mathbb{P}^1_k$ étale outside $\{0, 1, \infty\}$ and we obtain a genus 1 dessin $\mathcal{D} = \beta^{-1}([0,1])$. But since β is defined over a field containing $\mathbb{Q}(j)$ the element σ cannot act trivially on β and hence cannot act trivially on $\mathcal{D}$. The result follows. $\square$

The genus 0 case is more difficult. We employ a proof of [Schneps] which is similar to the procedure given in the proof of Belyi's theorem. We will freely use the following elementary properties of polynomials: for f a polynomial of degree n and d dividing n, if there exists some monic polynomial h with $h(0) = 0$ and $\mathbf{deg}(h) = d$, and some polynomial g such that $f = g \circ h$ then h is unique; moreover for g, h, g_0, and h_0 polynomials with $\mathbf{deg}(h) = \mathbf{deg}(h_0)$ and $g \circ h = g_0 \circ h_0$, then there exists constants c and d such that $h_0 = ch + d$.

In order to show that the action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of tree dessins is faithful, it suffices to show that for each nontrivial $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ there exists a Belyi function $\beta(z)$ corresponding to a tree such that $(\sigma \cdot \beta)(z) \neq \beta(\frac{az+b}{cz+d})$ unless $\frac{az+b}{cz+d} = z$. In particular if $(\sigma \cdot \beta)(z) = \beta(\frac{az+b}{cz+d})$ we must have $c = 0$ since $(\sigma \cdot \beta)(z)$ is a polynomial; we may then assume without loss of generality that $d = 1$ up to replacement of a and b, so it suffices to show that $(\sigma \cdot \beta)(z) \neq \beta(az + b)$ unless $a = 1$ and $b = 0$. We proceed as follows:

Theorem 3.12. *The action of* $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on the set of isomorphism classes of plane tree dessins is faithful.*

Proof. Let $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, let K be a number field, and let α be a primitive element for K such that the action of σ on α is nontrivial. We will construct a tree dessin over K on which σ acts nontrivially as follows: let $f_\alpha(z) \in K[z]$ be a polynomial with derivative $f'_\alpha(z) = z^3(z-1)^2(z-\alpha)$. By Belyi's theorem there exists a polynomial $f(z) \in \mathbb{Q}[z]$ such that $\beta_\alpha = f \circ f_\alpha$ is a Belyi polynomial; since f is defined over $\mathbb{Q}$ we obtain another Belyi polynomial $\beta_{\sigma\cdot\alpha} = f \circ f_{\sigma\cdot\alpha}$ where $f_{\sigma\cdot\alpha} = \sigma \cdot f_\alpha$.

Now let $\mathcal{D}_\alpha$ and $\mathcal{D}_{\sigma\cdot\alpha}$ be the tree dessins corresponding to the Belyi polynomials β_α and $\beta_{\sigma\cdot\alpha}$ respectively so that $\mathcal{D}_{\sigma\cdot\alpha} = \sigma \cdot \mathcal{D}_\alpha$. We wish to show that σ acts nontrivially on $\mathcal{D}_\alpha$ so that $\mathcal{D}_\alpha$ and $\mathcal{D}_{\sigma\cdot\alpha}$ are distinct; to that end it suffices to show that $\beta_{\sigma\cdot\alpha}(z) \neq \beta_\alpha(az + b)$ unless $a = 1$ and $b = 0$. To that end suppose $\beta_{\sigma\cdot\alpha}(z) = \beta_\alpha(ax + b)$ so that $f(f_{\sigma\cdot\alpha}(z)) = f(f_\alpha(az + b))$. Then by the previous lemma there exists unique c and d such that $f_\alpha(az + b) = cf_{\sigma\cdot\alpha}(z) + d$.

But then $cf_{\sigma\cdot\alpha}(z) + d$ has the same critical points as $f_{\sigma\cdot\alpha}(z)$, namely the points 0, 1, and $\sigma \cdot \alpha$, whereas $f_\alpha(az+b)$ has critical points x_0, x_1, and x_α with $ax_0 + b = 0$, $ax_1 + b = 1$, and $ax_\alpha + b = \alpha$. Since the function $az + b$ must take these critical points to the critical points of $f_\alpha(z)$, we have $x_0 = 0$, $x_1 = 1$, and $x_\alpha = \alpha$. But then the relations $ax_0 + b = 0$ and $ax_1 + b = 1$ implies $a = 1$ and $b = 0$, so the relation $ax_\alpha + b = \alpha$ implies $\sigma \cdot \alpha = \alpha$, contradicting the assumption that σ acts nontrivially on α. It follows that $\beta_{\sigma\cdot\alpha}(z) \neq \beta_\alpha(az+b)$ unless $a = 1$ and $b = 0$, and hence the dessins $\mathcal{D}_\alpha$ and $\sigma \cdot \mathcal{D}_\alpha = \mathcal{D}_{\sigma\cdot\alpha}$ are distinct. The result follows. □

Corollary 3.13. *The action of* $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on the set of isomorphism classes of genus* 0 *dessins is faithful.*

Put another way, both the sets of isomorphism classes of genus 0 dessins and genus 1 dessins, and even the set of isomorphism classes of plane tree dessins, is enough to study the structure of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; indeed every element $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is witnessed by some pair of dessins on which σ acts nontrivially, which is guaranteed to exist by faithfulness.

4 The Grothendieck-Teichmüller Group

We have seen that there is a natural $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on dessins which is arithmetic in the sense that the action is given by the Galois action on coefficients of Belyi functions defined over number fields. By contrast there is a geometric action of $\mathbf{Out}(\widehat{F}_2)$ on dessins given by geometric monodromy: any element $\sigma \in \mathbf{Out}(\widehat{F}_2)$ acts on the étale fundamental group $\widehat{\pi}_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}, \vec{01}) \simeq \widehat{F}_2$ and hence on finite étale covers of $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}$ and on isomorphism classes of dessins. Moreover each $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ induces via the outer Galois representation $\rho_{\text{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F}_2)$ an element $\rho_{\text{out}}(\sigma) \in \mathbf{Out}(\widehat{F}_2)$ acting by geometric monodromy.

What is perhaps surprising is that the elements $\rho_{\text{out}}(\sigma) \in \mathbf{Out}(\widehat{F}_2)$ acting by geometric monodromy already determine the action of $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; in other words the outer Galois representation $\rho_{\text{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F}_2)$ is faithful. This invites the following approach: since $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is realized via the outer Galois representation as a profinite subgroup of $\mathbf{Out}(\widehat{F}_2)$, it may be possible to characterize the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in terms of certain relations on its image in $\mathbf{Out}(\widehat{F}_2)$. We will realize the first step of this: we will exhibit two relations called the 2-cycle and 3-cycle relations which make the

4.1 Outer Galois Representation

In order to clarify this situation let $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$ and consider the homotopy exact sequence associated to X with respect to some $\mathbb{Q}$-rational tangential basepoint $\vec{v}$;

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{F}_2 & \longrightarrow & \widehat{\pi}_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}, \vec{01}) & \overset{s_{\vec{v}}}{\longleftrightarrow} & \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \overset{\rho_{\vec{v}}}{\swarrow} & \downarrow \rho_{\text{out}} & & \\
0 & \longrightarrow & \mathbf{Inn}(\widehat{F}_2) & \longrightarrow & \mathbf{Aut}(\widehat{F}_2) & \longrightarrow & \mathbf{Out}(\widehat{F}_2) & \longrightarrow & 0
\end{array}
$$

By definition for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\widetilde{\sigma} \in \widehat{\pi}_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}, \vec{v})$ a choice of lift of σ the outer Galois representation is given on $\gamma \in \widehat{\pi}_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}, \vec{v}) \simeq \widehat{F}_2$ by $\rho_{\text{out}}(\sigma)(\gamma) = \widetilde{\sigma} \circ \gamma \circ \widetilde{\sigma}^{-1}$, where γ is viewed as an element of $\widehat{\pi}_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}, \vec{v})$ under the canonical inclusion. Of course this depends on the choice of lifting $\widetilde{\gamma}$, but it only depends up to inner automorphism and hence we obtain the outer Galois representation $\rho_{\text{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F}_2)$ as discussed previously.

What is surprising is that the outer Galois representation $\rho_{\text{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F}_2)$ turns out to be faithful. One way to see this is by the action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on elliptic curves: if this were not faithful then there would be some possibly infinite extension $\overline{\mathbb{Q}}/L$ such that every elliptic curve defined over $\overline{\mathbb{Q}}$ is defined over L, a contradiction. More precisely:

Theorem 4.1. *The outer Galois representation $\rho : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{F_2})$ is faithful.*

Proof. Let $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$. Suppose ρ has nontrivial kernel fixing a possibly infinite extension L of $\mathbb{Q}$ so that the representation $\rho|_L : \mathbf{Gal}(L/\mathbb{Q}) \to \mathbf{Out}(\widehat{\pi}_1(X_{\overline{\mathbb{Q}}}))$ is trivial. Then for X_L the base change of X to L we have the short exact sequence

$$0 \to \widehat{\pi}_1(X_{\overline{\mathbb{Q}}}) \to \widehat{\pi}_1(X_L) \to \mathbf{Gal}(\overline{\mathbb{Q}}/L) \to 0$$

But $\widehat{\pi}_1(X_{\overline{\mathbb{Q}}})$ is a free profinite group hence has trivial centralizer in $\widehat{\pi}_1(X_L)$, and so the above short exact sequence splits. It follows that the base change functor $\mathbf{FEt}(X_L) \to \mathbf{FEt}(\overline{X})$ is essentially surjective, and hence by Belyi it follows that any integral proper normal curve over $\overline{\mathbb{Q}}$ is defined over L. But this is impossible: let E be an elliptic curve over $\overline{\mathbb{Q}}$ such that $j(E) \notin L$, and let E be an elliptic curve over L such that $E \simeq E_{\overline{\mathbb{Q}}}$. Then the Jacobian J is an elliptic curve over L such that $E_{\overline{\mathbb{Q}}} \simeq J_{\overline{\mathbb{Q}}}$ hence $j(\overline{E}) = j(J) \in L$, a contradiction. It follows that ρ has trivial kernel hence is faithful. □

Remark 4.1. *Belyi's theorem on the outer Galois representation is an example of a much more general theorem [Belyi-Matsumoto] on anabelian varieties. Specifically, for every affine hyperbolic curve X over a number field k, the outer Galois representation $\mathbf{Gal}(\overline{k}/k) \to \mathbf{Out}(\widehat{\pi}_1(X_{\overline{k}}, \overline{x}))$ is faithful. Of course $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ is the simplest among these: any other affine hyperbolic curve either has more punctures or greater genus.*

Of course this proof is dissatisfying as it does not give an explicit description of the inclusion $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathbf{Out}(\widehat{F}_2)$, nor does it indicate how to extend the Galois action to an action of $\mathbf{Out}(\widehat{F}_2)$. In order to describe the outer Galois representation more explicitly, we need a more useful description of the profinite groups $\widehat{F}_2$ and $\mathbf{Out}(\widehat{F}_2)$. This amounts to choosing a manageable inverse system of finite groups $\{H_n\}_{n\geq 1}$ with inverse limit $\varprojlim_{n\in\mathbb{N}} H_n \simeq \widehat{F}_2$ which are characteristic so as to obtain an isomorphism $\varinjlim_{n\in\mathbb{N}} \mathbf{Out}(H_n) \simeq \mathbf{Out}(\widehat{F}_2)$. There are many inverse systems of finite groups which achieve this, the simplest being the characteristic quotients of F_2:

Definition 4.1. *Let F_2 be a group and let $F_2^{(n)}$ denote the intersection of all normal subgroups N of F_2 with index $[F_2 : N] \leq n$. Then the characteristic quotient of F_2 is the quotient $H_n = F_2/F_2^{(n)}$.*

Clearly each H_n is a finite group, and moreover we have $H_n^{(n)} = 0$ for $n \geq 1$. Moreover the finite groups H_n are universal with respect to these properties: for each finite group G and each $g, h \in G$ there exists a morphism $\varphi : H_n \to G$ such that $\varphi(x) = g$ and $\varphi(y) = h$. Moreover for each group G with $G^{(n)} = 0$ and generators $g, h \in G$ there exists an epimorphism $\varphi : H_n \to G$ such that $\varphi(x) = g$ and $\varphi(y) = h$. In particular for $x', y' \in H_n$ generators there exists an automorphism $\varphi : H_n \to H_n$ such that $\varphi(x) = x'$ and $\varphi(y) = y'$. Moreover there exists a surjective morphism $H_{n+1} \to H_n$ with kernel the characteristic subgroup $H_{n+1}^{(n)}$, and we have morphisms $\mathbf{Aut}(H_{n+1}) \to \mathbf{Aut}(H_n)$ and $\mathbf{Out}(H_{n+1}) \to \mathbf{Out}(H_n)$.

The characteristic quotients H_n are related to dessins as follows. Consider the groupoid $\mathcal{G}_n$ whose objects are tuples (G, g, h) with G a finite group of order $\#G \leq n$ and $g, h \in G$, and whose isomorphisms $(G_0, g_0, h_0) \to (G_1, g_1, h_1)$ are isomorphisms $\varphi : G_0 \to G_1$ such that $\varphi(g_0) = g_1$ and $\varphi(h_0) = h_1$. Choose representatives for each isomorphism class in $\mathcal{G}_n$, say $(G_1, x_1, y_1), ..., (G_N, x_N, y_N)$, which amounts to classifying isomorphism classes of Galois dessins of order at most n. Now consider the group $G = G_1 \times ... \times G_n$ and the elements $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$. Then the subgroup $H = \langle x, y\rangle$ of G is isomorphic to H_n since H is finite and $H^{(n)} = 0$, so the canonical morphism $\varphi : H_n \to H$ is an isomorphism.

Indeed the characteristic quotients H_n form an inverse system of finite groups whose inverse limit is $\widehat{F}_2$. As such we will be able to describe $\widehat{F}_2$ and its automorphism groups $\mathbf{Aut}(\widehat{F}_2)$ and $\mathbf{Out}(\widehat{F}_2)$ in terms of the finite groups H_n which are much easier to control.

Lemma 4.2. *We have an isomorphism $\widehat{F}_2 \simeq \varprojlim_{n\in\mathbb{N}} H_n$.*

Proof. By definition $\widehat{F}_2$ is the limit of quotients F_2/N taken over normal subgroups N of finite index in F_2. But each such normal subgroup contains the characteristic subgroup $F_2^{(n)}$ for some $n \in \mathbb{N}$, and hence the diagram $\{F_2^{(n)}\}_{n\in\mathbb{N}}$ is final. It follows that

$$\widehat{F}_2 = \varprojlim_{N\leq F_2} F_2/N \simeq \varprojlim_{n\in\mathbb{N}} F_2/F_2^{(n)} = \varprojlim_{n\in\mathbb{N}} H_n \qquad \square$$

Now recall that we have a bijection between normal subgroups of finite index in F_2 and open normal subgroups of $\widehat{F}_2$, and that every open normal subgroup of $\widehat{F}_2$ is by definition closed and of finite index. Indeed for each normal subgroup N of finite index in F_2, the quotient map $F_2 \to F_2/N$ extends to a map $\widehat{F}_2 \to F_2/N$ with kernel the closure of N in $\widehat{F}_2$. In particular the closure of the characteristic subgroup $F_2^{(n)}$ in $\widehat{F}_2$ is preserved by all continuous automorphisms of $\widehat{F}_2$. We will use a theorem of [Jarden] that any automorphism of $\widehat{F}_2$ with fixes all open normal subgroups is an inner automorphism.

Theorem 4.3. *We have an isomorphism* $\mathbf{Out}(\widehat{F}_2) \simeq \varprojlim_{n\in\mathbb{N}} \mathbf{Out}(H_n)$.

Proof. In view of the isomorphism $\widehat{F}_2 \simeq \varprojlim_{n\in\mathbb{N}} H_n$ consider the canonical morphism

$$\varprojlim_{n\in\mathbb{N}} \mathbf{Aut}(H_n) \to \mathbf{Aut}(\varprojlim_{n\in\mathbb{N}} H_n) \simeq \mathbf{Aut}(\widehat{F}_2)$$

with kernel a characteristic subgroup of $\varprojlim_{n\in\mathbb{N}} \mathbf{Aut}(H_n)$ so that the above morphism has an inverse $\mathbf{Aut}(\widehat{F}_2) \simeq \mathbf{Aut}(\varprojlim_{n\in\mathbb{N}} H_n) \to \varprojlim_{n\in\mathbb{N}} \mathbf{Aut}(H_n)$. We claim that we have a surjective morphism

$$\pi : \varprojlim_{n\in\mathbb{N}} \mathbf{Aut}(H_n) \to \varprojlim_{n\in\mathbb{N}} \mathbf{Out}(H_n)$$

is surjective. To see this consider an element $\{\varphi_n \in \mathbf{Out}(H_n)\}_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathbf{Out}(H_n)$ and choose representatives $\{\widetilde{\varphi} \in \mathbf{Aut}(H_n)\}_{n\in\mathbb{N}}$. Now it need not be the case that $\widetilde{\varphi}_{n+1}$ maps to $\widetilde{\varphi}_n$ under the canonical projection $\mathbf{Aut}(H_{n+1}) \to \mathbf{Aut}(H_n)$, but this holds up to composition with an inner automorphism of H_{n+1}, and hence the representatives define an element $\widetilde{\varphi} = \{\widetilde{\varphi}_n \in \mathbf{Aut}(H_n)\}_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathbf{Aut}(H_n) \simeq \mathbf{Aut}(\widetilde{F}_2)$ mapping to φ via π, so it follows that π is surjective.

Now it suffices to show that π has kernel $\mathbf{Inn}(\widehat{F}_2)$. To that end we note that any element of $\mathbf{ker}(\pi)$ fixes each open normal subgroup of finite index in $\widehat{F}_2$ since each such open normal subgroup of finite index is the closure $\overline{N}$ of a normal subgroup N of finite index in F_2, and each such subgroup N contains $F_2^{(n)}$ for n sufficiently large. But then each element of $\mathbf{ker}(\pi)$ induces an inner automorphism of H_n and hence fixes $\overline{N}$. It follows by [Jarden] that $\mathbf{ker}(\pi) = \mathbf{Inn}(\widehat{F}_2)$ and in view of the short exact sequence

$$0 \to \mathbf{Inn}(\widehat{F}_2) \to \mathbf{Aut}(\widehat{F}_2) \to \mathbf{Out}(\widehat{F}_2) \to 0$$

it follows that $\varprojlim_{n\in\mathbb{N}} \mathbf{Out}(H_n) \simeq \mathbf{Out}(\widehat{F}_2)$. $\square$

Having described the characteristic quotients H_n and their relation to the outer automorphism group $\mathbf{Out}(\widehat{F}_2)$, we can now compute the action of $\mathbf{Out}(\widehat{F}_2)$ on dessins. Fix an algebraic closure Ω of $\overline{\mathbb{Q}}(t)$. By definition finite group H_n yields a regular dessin $\mathcal{C}$ and a Galois extension $L_n/\overline{\mathbb{Q}}(t)$ unramified over $\{0, 1, \infty\}$ with Galois group $\mathbf{Gal}(L_n/\overline{\mathbb{Q}}(t)) \simeq H_n$; we may assume that L_n is a subfield of Ω in which case L_n is unique up to isomorphism.

Now for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the dessin $\sigma(X)$ is also regular, corresponding to a choice of generators $\sigma(x), \sigma(y) \in H_n$. But then by the universal property of H_n there exists an automorphism $\varphi : H_n \to H_n$ such that $\varphi(x) = \sigma(x)$ and $\varphi(y) = \sigma(y)$, and hence the fields L_n and $\sigma(L_n)$ are isomorphic; in other words we have an isomorphism $\iota : L_n \xrightarrow{\sim} \sigma(L_n)$ which is well defined up to precomposition with an element of $\mathbf{Gal}(L_n/\overline{\mathbb{Q}}(t)) \simeq H_n$. But then for $h \in H_n$ the map $\iota^{-1} \circ \sigma(h) \circ \iota : L_n \xrightarrow{\sim} \sigma(L_n) \to \sigma(L_n) \xrightarrow{\sim} L_n$ is well defined up to conjugation by an element of H_n, and hence the automorphism $\rho_n(\sigma)(h) = \iota^{-1} \circ \sigma(h) \circ \iota$ of H_n yields a well defined element $\rho_n(\sigma) \in \mathbf{Out}(H_n)$ depending only on σ. The resulting map $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(H_n)$ is the required Galois representation at each of the characteristic quotients H_n, and taking the inverse limit over $n \geq 1$ we obtain a Galois representation $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \varprojlim_{n \geq 1} \mathbf{Out}(H_n) \simeq \mathbf{Out}(\widehat{F}_2)$ which is the outer Galois representation from before.

Theorem 4.4. *The outer Galois representation $\rho : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \varprojlim_{n \in \mathbb{N}} \mathbf{Out}(H_n)$ given by $\rho(\sigma)_n = \rho_n(\sigma)$ is faithful.*

Proof. Consider the group homomorphism $\rho_n : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(H_n)$ associating to each $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the automorphism $\rho_n(\sigma) \in \mathbf{Out}(H_n)$ as above. We must show that the morphisms $\rho_n : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(H_n)$ are compatible with the canonical projections $\mathbf{Out}(H_{n+1}) \to \mathbf{Out}(H_n)$. To that end we note that the extension L_{n+1}/L_n is characteristic in the sense that $\mathbf{Gal}(L_{n+1}/L_n) \simeq H_{n+1}^{(n)}$ is a characteristic subgroup of H_{n+1} and hence any isomorphism $L_{n+1} \xrightarrow{\sim} \sigma \cdot L_{n+1}$ sends L_n to $\sigma \cdot L_n$. But then by naturality of the isomorphism σ it follows that the composition $\rho_{n+1} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(H_{n+1}) \to \mathbf{Out}(H_n)$ is equal to the morphism $\rho_n : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(H_n)$ as claimed.

It remains to show that $\rho : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \varprojlim_{n \in \mathbb{N}} \mathbf{Out}(H_n)$ is injective. Since the action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins is faithful it suffices to show that for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\rho(\sigma) = 1$ then the action of σ on dessins is trivial. To that end choose an extension L of $\overline{\mathbb{Q}}(t)$ unramified outside $\{0, 1, \infty\}$, which by definition corresponds to a subgroup K of L_n for $n \geq 1$ sufficiently large. Then $\sigma \cdot L$ corresponds to $\sigma \cdot K$ as a subfield of $\sigma \cdot L_n$, and since $\rho(\sigma) = 1$ we may choose an isomorphism $L_n \xrightarrow{\sim} \sigma \cdot L_n$ so that $\sigma \cdot L$ corresponds to a conjugate subgroup of K in which case $\sigma \cdot L$ is isomorphic to L. The result follows. ☐

One benefit of using the characteristic quotients H_n and their relationship to dessins is that the faithful outer action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on each H_n and hence on dessins extends to an action of $\varinjlim_{n \in \mathbb{N}} \mathbf{Out}(H_n) \simeq \mathbf{Out}(\widehat{F}_2)$ on dessins via the outer Galois representation $\rho_{\text{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathbf{Out}(\widehat{F}_2)$.

Let $\mathcal{D}$ be a (connected) dessin represented by a quotient G/K with G a finite group with two generators $x, y \in G$ and K a subgroup of G. Now if G has order at most n we have a surjective map $p : H_n \to G$ sending $x, y \in H_n$ to $x, y \in G$ respectively. Write $N = \mathbf{ker}(p)$ and $\widetilde{K} = p^{-1}(K)$ for the respective subgroups of H_n. Given an automorphism $\sigma \in \mathbf{Aut}(H_n)$ we consider the quotient $\sigma \cdot G = H_n/\sigma \cdot N$ with generators [3] the images of $x, y \in H_n$ under the canonical projection $H_n \to H_n/\sigma \cdot N$. But then the image $\sigma \cdot K$ of $\sigma \cdot \widetilde{K}$ under the canonical projection $H_n \to H_n/\sigma \cdot N$ is a subgroup of $\sigma \cdot G$, in which case the quotient $\sigma \cdot G/\sigma \cdot K$ represents a (connected) dessin $\sigma \cdot \mathcal{D}$. By construction this yields an action of $\mathbf{Out}(H_n)$ on isomorphism classes of (connected) dessins with cartographic group G of order at most n.

[3] One is tempted to take $\sigma \cdot x$ and $\sigma \cdot y$ as generators for $H_n/\sigma \cdot N$, but this does not work since the group isomorphism $\sigma : G \xrightarrow{\sim} \sigma \cdot G$ is not compatible with the distinguished generators $x, y \in G$.

Lemma 4.5. *The above action of $\mathbf{Out}(H_n)$ on isomorphism classes of connected dessins of order at most n extends to a compatible action of $\varprojlim_{n\in\mathbb{N}} \mathbf{Out}(H_n) \simeq \mathbf{Out}(\widehat{F}_2)$ on isomorphism classes of dessins. Moreover for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the action of σ on dessins coincides with that of $\rho(\sigma) \in \mathbf{Out}(\widehat{F}_2)$.*

Proof. For the first claim, it suffices to show that for all $n, m \geq 1$ and for all $\sigma_{n+m} \in \mathbf{Out}(H_{n+m})$ with $\sigma_n \in \mathbf{Out}(H_n)$ its image under the canonical projection $\mathbf{Out}(H_{n+m}) \to \mathbf{Out}(H_n)$, then for any dessin $\mathcal{D}$ with cartographic group G of order at most n; the dessins $\sigma_{n+m} \cdot \mathcal{D}$ and $\sigma_n \cdot \mathcal{D}$ are isomorphic. But this is immediate since the canonical projection $p_{n+m} : H_{n+m} \to G$ factors as $p_n \circ q_{n+m}$ where $p_n : H_n \to G$ and $q_{n+m} : H_{n+m} \to H_n$ are the canonical projections.

For the second claim, let $\mathcal{C}$ be the dessin corresponding to the finite group H_n. Then for G a finite group of order at most n with two generators $x, y \in G$ and H a subgroup of G, the quotient G/K represents a dessin $\mathcal{D}$ subordinate to $\mathcal{C}$ corresponding to the subgroup $\widetilde{K}$ as above; then for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the dessin $\sigma \cdot \mathcal{D}$ corresponds to the subgroup $\sigma \cdot (\widetilde{K})$ of $\mathbf{Aut}(\sigma \cdot \mathcal{C}) \simeq \sigma \cdot H_n$. But then by choosing an isomorphism $\iota : \mathcal{C} \xrightarrow{\sim} \sigma \cdot \mathcal{C}$ it follows that $\sigma \cdot \mathcal{D}$ is isomorphic to $H_n/\sigma \cdot \widetilde{K}$ as claimed. □

Theorem 4.6. *The action of $\mathbf{Out}(\widehat{F}_2)$ on isomorphism classes of Galois dessins is faithful.*

Proof. It suffices to show this for the set of isomorphism classes of Galois dessins. Let $\varphi \in \mathbf{Aut}(\widehat{F}_2)$ represent an element $[\varphi] \in \mathbf{Out}(\widehat{F}_2)$. If the action of $[\varphi]$ is trivial on the set of isomorphism classes of regular dessins, then φ must fix all open normal subgroups of $\widehat{F}_2$. But then by [Jarden] it follows that $\varphi \in \mathbf{Inn}(\widehat{F}_2)$ and hence $[\varphi] = 1$. The result follows. □

Corollary 4.7. *The action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on isomorphism classes of Galois dessins is faithful.*

In nice cases we can easily compute the action of $\sigma \in \mathbf{Out}(\widehat{F}_n)$ on dessins when σ is represented in terms of its action on the generators of each H_n. The most important examples are the automorphisms $\theta, \omega \in \mathbf{Out}(\widehat{F}_2)$ sending a dessin to its recoloring and its dual. The first is easiest to see by pulling back generators:

Example 4.1. *Let θ be an automorphism of H_n given on the generators $x, y \in H_n$ by $\theta(x) = y$ and $\theta(y) = x$. Let $\mathcal{D}$ be a Galois dessin of order at most n represented by a finite group of order at most n with two distinguished generators $x, y \in G$. Write $G = H_n/N$ for some unique normal subgroup N of H_n and consider the canonical morphism $G = H_n/N \to H_n/\theta(N)$ induced by θ. Pulling back the canonical generators $x, y \in H_n/\theta(N)$ we obtain $\theta^{-1}(x), \theta^{-1}(y) \in G$ as generators. Then since $\theta^{-1}(x) = y$ and $\theta^{-1}(y) = x$ it follows that $\theta \cdot (G, x, y) = (G, y, x)$ and hence θ sends the dessin $\mathcal{D}$ to its recoloring as claimed.*

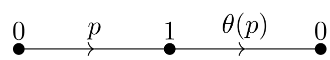
The second is similar, although we must be more clever defining $\omega(x)$ and $\omega(y)$ so that the pullbacks $\omega^{-1}(x)$ and $\omega^{-1}(y)$ yield the correct generators.

Example 4.2. *Let ω be an automorphism of H_n given on the generators $x, y \in H_n$ by $\omega(x) = yzy^{-1}$ and $\omega(y) = y$. Let $\mathcal{D}$ be a Galois dessin of order at most n represented by a finite group G of order at most n with two distinguished generators $x, y \in G$. Write $G = H_n/N$ for some unique normal subgroup N of H_n and consider the canonical morphism $G = H_n/N \to H_n/\omega(N)$ induced by ω. Pulling back the canonical generators $x, y \in H_n/\omega(N)$ we obtain $\omega^{-1}(x), \omega^{-1}(y) \in G$ as generators. Then since $\omega^{-1}(x) = z$ and $\omega^{-1}(y) = y$ it follows that $\omega \cdot (G, x, y) = (G, z, y)$ and hence ω sends the dessin $\mathcal{D}$ to its dual as claimed.*

4.2 The Coarse Grothendieck-Teichmüller Group

It is perhaps surprising that the outer Galois representation $\rho_{\text{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathbf{Out}(\widehat{F}_2)$ is faithful: whereas the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is necessarily defined in terms of its action on number fields, the outer automorphism group $\mathbf{Out}(\widehat{F}_2)$ is a purely group-theoretic object and makes no reference to the Galois theory of number fields. For this reason one is tempted to ask how $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ can be characterized by its image in $\mathbf{Out}(\widehat{F}_2)$ or its lift to $\mathbf{Aut}(\widehat{F}_2)$. The coarse Grothendieck Teichmüller group $\widehat{\mathbf{GT}}_0$ is the first step to realizing this program: we can deduce certain 2-cycle and 3-cycle relations among the elements in the image of $\rho_{\text{out}} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathbf{Out}(\widehat{F}_2)$ coming from geometric monodromy which defines an inclusion of profinite groups $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}}_0$ which is compatible with the cyclotomic chraracter $\chi : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^\times$ and the Galois parameter $\mathfrak{f} : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to [\widehat{F}_2, \widehat{F}_2]$.

We have already seen the construction of the (tangential) étale fundamental groupoid $\widehat{\pi}_1(X; \mathcal{B})$ for $X = \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ with respect to the tangential basepoints $\mathcal{B} = \{\vec{01}, \vec{10}\}$ associated to the local coordinates $t_{\vec{01}} = t$ and $t_{\vec{10}} = 1 - t$; in this situation the involution $\theta(t) = 1 - t$ of X acts on $\mathcal{B}$ and hence on $\widehat{\pi}_1(X; \mathcal{B})$ by sending pro-loops $\gamma \in \widehat{\pi}_1(X, \vec{v})$ to pro-loops $\theta(\gamma) \in \widehat{\pi}_1(X, \theta(\vec{v}))$, and by sending pro-paths $\gamma \in \widehat{\pi}_1(X; \vec{v}, \vec{w})$ to pro-paths $\theta(\gamma) \in \widehat{\pi}_1(X; \theta(\vec{v}); \theta(\vec{w}))$. In particular for $p \in \widehat{\pi}_1(X; \vec{01}, \vec{10})$ the straight path and $\theta(p) \in \widehat{\pi}_1(X; \vec{10}, \vec{01})$ the opposite path we have $\theta(p)p = 1$ since the associated loop $\theta(p)p \in \pi_1(X^{\mathbf{an}}, \vec{01})$ is nullhomotopic.



Moreover since the involution θ of X is $\mathbb{Q}$-rational it is $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant, so that for each $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\gamma \in \widehat{\pi}_1(X; \vec{v}, \vec{w})$ we have $\sigma \cdot \theta(\gamma) = \theta(\sigma \cdot \gamma) \in \widehat{\pi}_1(X; \theta(\vec{v}), \theta(\vec{w}))$. In particular by applying the Galois action to both sides of the identity $\theta(p)p = 1$ we obtain the 2-cycle relation for the Galois parameter $\mathfrak{f}_\sigma$.

Theorem 4.8. *(2-cycle relation) for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have*

$$\mathfrak{f}_\sigma(y, x)\mathfrak{f}_\sigma(x, y) = 1$$

Proof. Let $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and consider the action of σ on $p \in \widehat{\pi}_1(X; \vec{01}, \vec{10})$ given by $\sigma \cdot p = \sigma p \sigma^{-1} = p\mathfrak{f}_\sigma(x, y) \in \widehat{\pi}_1(X; \vec{01}, \vec{10})$. Since the involution θ of X is $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant, we have

$$\begin{aligned} \sigma \cdot \theta(p) = \theta(p)\mathfrak{f}_\sigma(\theta(x), \theta(y)) &= p^{-1}\mathfrak{f}_\sigma(pyp^{-1}, pxp^{-1}) \\ &= p^{-1}p\mathfrak{f}_\sigma(y, x)p^{-1} = \mathfrak{f}_\sigma(y, x)p^{-1} \end{aligned}$$

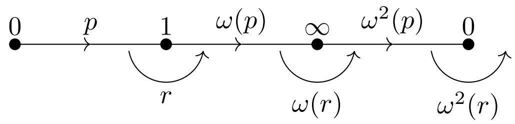
But then since $\theta(p)p = 1$ we have $(\sigma \cdot \theta(p))(\sigma \cdot p) = 1$, and we obtain the desired relation:

$$(\sigma \cdot \theta(p))(\sigma \cdot p) = \mathfrak{f}_\sigma(y, x)p^{-1}p\mathfrak{f}_\sigma(x, y) = \mathfrak{f}_\sigma(y, x)\mathfrak{f}_\sigma(x, y) = 1 \qquad \square$$

We can prove the 3-cycle relation by similar means, using a larger set of tangential basepoints $\mathcal{B} = \{\vec{01}, \vec{10}, \vec{0\infty}, \vec{\infty 0}, \vec{1\infty}, \vec{\infty 1}\}$ associated to the local coordinates

$$t_{\vec{01}} = t \qquad t_{\vec{10}} = 1 - t \qquad t_{\vec{0\infty}} = \frac{t}{t-1} \qquad t_{\vec{\infty 0}} = \frac{1}{1-t} \qquad t_{\vec{1\infty}} = \frac{t-1}{t} \qquad t_{\vec{\infty 1}} = \frac{1}{t}$$

In this situation the automorphism $\omega(t) = \frac{1}{1-t}$ of X acts on $\mathcal{B}$ and hence on $\widehat{\pi}_1(X;\mathcal{B})$ by sending pro-loops $\gamma \in \widehat{\pi}_1(X,\vec{v})$ to pro-loops $\omega(\gamma) \in \widehat{\pi}_1(X;\omega(\vec{v}))$, and sending pro-paths $\gamma \in \widehat{\pi}_1(X;\vec{v},\vec{w})$ to pro-paths $\omega(\gamma) \in \widehat{\pi}_1(X;\omega(\vec{v}),\omega(\vec{w}))$. In particular for $p \in \widehat{\pi}_1(X;0\vec{1},1\vec{0})$ the straight path, $r \in \widehat{\pi}_1(X;1\vec{0},1\vec{\infty})$ the half loop around 1, and $q = pr \in \widehat{\pi}_1(X;0\vec{1},1\vec{\infty})$ their composition, we have $\omega^2(q)\omega(q)q = 1$ since the associated loop $\omega^2(q)\omega(q)q \in \pi_1(X^{\mathbf{an}},0\vec{1})$ is nullhomotopic.



Moreover since the automorphism ω of X is $\mathbb{Q}$-rational it is $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant, so that for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\gamma \in \widehat{\pi}_1(X;\vec{v},\vec{w})$ we have $\sigma \cdot \omega(\gamma) = \omega(\sigma \cdot \gamma) \in \widehat{\pi}(X;\omega(\vec{v}),\omega(\vec{w}))$. In particular by applying the Galois action to both sides of the identity $\omega^2(q)\omega(q)q = 1$ we obtain the 3-cycle relation for $\mathfrak{f}_\sigma$; for this we only need to calculate the action of σ on the half loop r, in which case the 3-cycle relation follows from the action on $q = rp$.

Lemma 4.9. *For $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the Galois action on $r \in \widehat{\pi}_1(X(\overline{\mathbb{Q}}),0\vec{1},1\vec{\infty})$ is given*

$$\sigma \cdot r = r\theta(x)^{\frac{\chi(\sigma)-1}{2}}$$

Proof. Consider $f \in \mathcal{M}_{1\vec{0}}$ so that $f = \sum_{i\geq -n} a_n t_{1\vec{0}}^{i/N} = \sum_{i\geq -n} a_n(1-t_{0\vec{1}})^{i/N}$ for $a_i \in \overline{\mathbb{Q}}$. Writing $f = \sum_{i\geq -n} a_i \exp(\frac{i}{N}\log(t_{1\vec{0}}))$, the analytic continuation of f along the half-loop r is given

$$r \cdot f = \sum_{i\geq -n} a_i \exp(\tfrac{i}{N}\log(t_{0\vec{1}}-1) + \pi\sqrt{-1}) = \sum_{i\geq -n} a_i \zeta_{N/2}^{i} \exp(\tfrac{i}{N}\log(t_{0\vec{1}}-1))$$

Then $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $r \cdot f$ by first applying σ^{-1} to a_i in f, then analytically continuing along r as above, then applying σ to $\sigma^{-1}(a_i)$ and the other terms. We obtain

$$(\sigma r \sigma^{-1})(f) = \sum_{i\geq -n} a_i \zeta_{N/2}^{i\chi(\sigma)} \exp(\tfrac{i}{N}\log(t_{0\vec{1}}-1))$$

Then by analytically continuing to the above expression along r^{-1} we incur a factor of $\zeta_{N/2}^{-i}$ and we obtain

$$(r^{-1}\sigma r \sigma^{-1}) \cdot f = \sum_{i\geq -n} a_i \zeta_N^{i\frac{\chi(\sigma)-1}{2}} \exp(\tfrac{i}{N}\log(1-t_{0\vec{1}})) = \sum_{i\geq -n} a_i \zeta_N^{i\frac{\chi(\sigma)-1}{2}} t_{1\vec{0}}^{i/N}$$

But since $\theta(t_{0\vec{1}}) = t_{1\vec{0}}$ the result follows by comparing the coefficients of $(r^{-1}\sigma r \sigma^{-1}) \cdot f$ to those of $\theta(x) \cdot f$ which are given $\theta(x) \cdot f = \sum_{i\geq -n} a_i \zeta_N^i t_{1\vec{0}}^{i/N}$. □

Theorem 4.10. *(3-cycle relation) For $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have*

$$\mathfrak{f}_\sigma(z,x) z^{\frac{\chi(\sigma)-1}{2}} \mathfrak{f}_\sigma(y,z) y^{\frac{\chi(\sigma)-1}{2}} \mathfrak{f}_\sigma(x,y) x^{\frac{\chi(\sigma)-1}{2}} = 1$$

Proof. For $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have by the previous lemma that $\sigma \cdot q = q\theta(x)^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(x,y)$. Since the automorphism ω of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant, and since $q^{-1}\omega(x)q = y$, $q^{-1}\omega(y)q = z$, and $q^{-1}\omega(z)q = x$ with $xyz = 1$, we obtain

$$\sigma \cdot \omega(q) = \omega(q)qz^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(y,z)$$
$$\sigma \cdot \omega^2(q) = \omega^2(q)\omega(q)qx^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(z,x)$$

But since $\omega^2(q)\omega(q)q = 1$ we have $\sigma \cdot (\omega^2(q)\omega(q)q) = (\sigma \cdot \omega^2(q))(\sigma \cdot \omega(q))(\sigma \cdot q) = 1$ so we obtain

$$x^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(z,x)z^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(y,z)y^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(x,y) = 1$$

But then we can cycle $x^{\frac{\chi(\sigma)-1}{2}}$ and obtain the desired relation:

$$\mathfrak{f}_\sigma(z,x)z^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(y,z)y^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(x,y)x^{\frac{\chi(\sigma)-1}{2}} = 1$$ □

Finally we have proved all the parts of the main theorem on the Galois representation in $\widehat{F}_2$. For convenience we state the whole theorem:

Corollary 4.11. *Each $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathbf{Aut}(\widehat{\pi}_1(X,\vec{01})) \simeq \mathbf{Aut}(\widehat{F}_2)$ is determined by parameters $(\chi(\sigma),\mathfrak{f}_\sigma) \in \widehat{\mathbb{Z}}^\times \times [\widehat{F}_2,\widehat{F}_2]$ where $\chi(\sigma) \in \widehat{\mathbb{Z}}^\times$ is the cyclotomic character and $\mathfrak{f}_\sigma \in [\widehat{F}_2,\widehat{F}_2]$ is the non-Abelian Galois symbol in the derived subgroup of $\widehat{F}_2$, acting on $\widehat{\pi}_1(X,\vec{01}) \simeq \widehat{F}_2 = \langle x,y\rangle^\wedge$ as*

$$\begin{cases} \sigma \cdot x = x^{\chi(\sigma)} \\ \sigma \cdot y = \mathfrak{f}_\sigma^{-1}y^{\chi(\sigma)}\mathfrak{f}_\sigma \end{cases}$$

Moreover the parameters $(\chi(\sigma),\mathfrak{f}_\sigma) \in \widehat{\mathbb{Z}}^\times \times \widehat{F}_2'$ satisfy the profinite 2-cycle and 3-cycle relations in $\widehat{F}_2$

$$\begin{cases} \mathfrak{f}_\sigma(y,x)\mathfrak{f}_\sigma(x,y) = 1 & (i) \\ \mathfrak{f}_\sigma(z,x)z^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(y,z)y^{\frac{\chi(\sigma)-1}{2}}\mathfrak{f}_\sigma(x,y)x^{\frac{\chi(\sigma)-1}{2}} = 1 & (ii) \end{cases}$$

Definition 4.2. *The coarse Grothendieck-Teichmüller group $\widehat{\mathbf{GT}}_0$ is the set of pairs $\sigma = (\lambda,f) \in \widehat{\mathbb{Z}}^\times \times [\widehat{F}_2,\widehat{F}_2]$ acting on $\widehat{F}_2 = \langle x,y\rangle^\wedge$ as above, satisfying the profinite relations in $\widehat{F}_2$*

$$\begin{cases} f(y,x)f(x,y) = 1 & (i) \\ f(z,x)z^{\frac{\lambda-1}{2}}f(y,z)y^{\frac{\lambda-1}{2}}f(x,y)x^{\frac{\lambda-1}{2}} = 1 & (ii) \end{cases}$$

With group operation on $(\lambda,f),(\mu,g) \in \widehat{\mathbf{GT}}_0$ given by

$$(\lambda,f)\circ(\mu,g) = \big(\lambda\mu, f(x^\mu, g^{-1}y^\mu g)\big)$$

Of course it is not obvious that this is a group, and indeed it is very difficult to show that the 2-cycle and 3-cycle relations are preserved by composition. In fact the easiest way to show that $\widehat{\mathbf{GT}}_0$ is through the much stronger characterization of $\widehat{\mathbf{GT}}_0$ as the profinite automorphism group of the first level of the so called genus 0 Teichmüller tower, which we discuss below. Taking this for granted, we finally obtain:

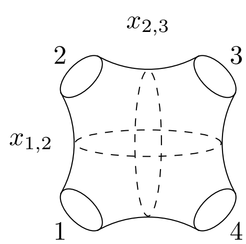
Corollary 4.12. *We have an injection of profinite groups* $\iota : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}}_0$ *such that the composition* $\chi : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}}_0 \to \widehat{\mathbb{Z}}^\times$ *is the cyclotomic character.*

Proof. Since for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the pair $(\chi(\sigma), \mathfrak{f}_\sigma)$ satisfies the same 2-cycle and 3-cycle relations defining $\widehat{\mathbf{GT}}_0$ it follows by Belyi's theorem that the map $\iota : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbf{GT}}_0$ given by $\iota(\sigma) = (\chi(\sigma), \mathfrak{f}_\sigma)$ is injective, and since for $\sigma, \tau \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the composition $(\chi(\sigma), \mathfrak{f}_\sigma) \circ (\chi(\tau), \mathfrak{f}_\tau) = \big(\chi(\sigma)\chi(\tau), \mathfrak{f}_\sigma(x^{\chi(\tau)}, \mathfrak{f}_\tau^{-1} y^{\chi(\tau)} \mathfrak{f}_\tau)\big)$ agrees with the group law for $\widehat{\mathbf{GT}}_0$ it follows that this is a morphisms of profinite groups. $\square$

Indeed it is easy to see that the 2-cycle and 3-cycle relations are effectively the only relations coming from the Galois representation $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{\pi}_1(X, \vec{01})) \simeq \mathbf{Aut}(\widehat{F}_2)$, since every nullhomotopic path in the fundamental groupoid $\widehat{\pi}_1(X; \mathcal{B})$ can be written in terms of the nullhomotopic paths $\theta(p)p, \omega^2(p)\omega(p)p \in \widehat{\pi}(X; \mathcal{B})$ and their orbits under $\mathbf{Aut}(X)$. Consequently in order to define further relations which characterize the image of the inclusion $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}}_0$, one must consider Galois representations in étale fundamental groups other than $\widehat{\pi}_1(X, \vec{01}) \simeq \widehat{F}_2$.

4.3 The Grothendieck-Teichmüller Group

The crucial observation made by Grothendieck is that in view of the isomorphism $\mathcal{M}_{0,4/\overline{\mathbb{Q}}} \simeq \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$, one should view the Galois representation $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{F}_2)$ as a Galois representation $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{\Gamma}_{0,4})$ where $\widehat{\pi}_1(\mathcal{M}_{0,4/\overline{\mathbb{Q}}}, \vec{v}) \simeq \widehat{\Gamma}_{0,4}$ is the profinite completion of the mapping class group $\Gamma_{0,4} \simeq F_2 = \langle x_{1,2}, x_{2,3} \rangle$. Indeed we can view the generators $x, y \in \widehat{F}_2$ as Dehn twists on a genus 0 curve with 4 ordered marked points



Indeed the three points at infinity of $\mathcal{M}_{0,4/\overline{\mathbb{Q}}} \simeq \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ are in bijection with the 3 distinct pants decompositions of type $(0, 4)$, namely those yielding the partitions $(12)(34)$, $(13)(24)$, and $(14)(23)$ of ordered marked points. Moreover S_4 acts canonically on these 4 marked points, and the above three partitions generate the Klein 4 group K in S_4, with the quotient $S_4/K \simeq S_3$ recovering the automorphism group of $\mathcal{M}_{0,4/\overline{\mathbb{Q}}}$.

This suggests considering the Galois representations $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{\Gamma}_{g,n})$ induced by tangential basepoints $\vec{v}$ of $\mathcal{M}_{g,n}$ corresponding to maximal degenerations of curves of type (g, n), where $\widehat{\pi}_1(\mathcal{M}_{g,n/\overline{\mathbb{Q}}}, \vec{v}) \simeq \widehat{\Gamma}_{g,n}$ is the profinite completion of the mapping class group $\widehat{\Gamma}_{g,n}$.

Of course the first thing to check is that $\widehat{\pi}_1(\mathcal{M}_{g,n/\overline{\mathbb{Q}}}, \vec{v}) \simeq \widehat{\Gamma}_{g,n}$, which takes some effort. Indeed the $\mathcal{M}_{g,n/\overline{\mathbb{Q}}}$ are not schemes but rather Deligne-Mumford stacks, and one must check that the relevant machinery and comparison theorems for étale fundamental groups carry over in this case. Indeed the theory of étale fundamental groups can be extended to algebraic stacks [Friedlander], and in particular by [Oda] the moduli spaces $\mathcal{M}_{g,n}(\mathbb{Q})$ admit a homotopy exact sequence, which is split given a $\mathbb{Q}$-rational basepoint or more generally a $\mathbb{Q}$-rational tangential basepoint $\overline{v}$ of $\mathcal{M}_{g,n}(\mathbb{Q})$

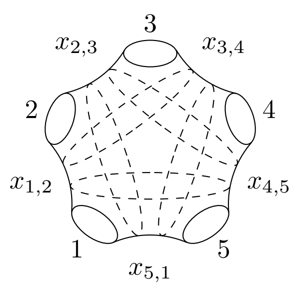
$$0 \longrightarrow \widehat{\Gamma}_{g,n} \longrightarrow \widehat{\pi}_1(\mathcal{M}_{g,n/\overline{\mathbb{Q}}}, \vec{v}) \underset{s_{\vec{v}}}{\overset{\curvearrowleft}{\longrightarrow}} \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 0$$

As before we obtain an outer Galois representation $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{\Gamma}_{g,n})$ which in the presence of a $\mathbb{Q}$-rational tangential basepoint yields a Galois representation $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{\Gamma}_{g,n})$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\Gamma}_{g,n} & \longrightarrow & \widehat{\pi}_1(\mathcal{M}_{g,n/\overline{\mathbb{Q}}}, \vec{v}) & \overset{s_{\vec{v}}}{\rightleftarrows} & \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \swarrow & \downarrow \rho_{\mathbf{out}} & & \\
0 & \longrightarrow & \mathbf{Inn}(\widehat{\Gamma}_{g,n}) & \longrightarrow & \mathbf{Aut}(\widehat{\Gamma}_{g,n}) & \longrightarrow & \mathbf{Out}(\widehat{\Gamma}_{g,n}) & \longrightarrow & 0
\end{array}$$

At least in the genus 0 case it follows by Belyi's theorem that the outer Galois representation $\rho_{\mathbf{out}}$: $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Out}(\widehat{\Gamma}_{0,n})$ is faithful, and moreover each of these projects onto the outer Galois representation on $\mathcal{M}_{0,4/\overline{\mathbb{Q}}}$. The goal is then to deduce interesting relations among these Galois representations from which we can deduce further relations among the elements of $\widehat{\mathbf{GT}}_0$.

The first interesting case of this is the Galois representation $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{Aut}(\widehat{\Gamma}_{0,5})$ from which we can deduce a third relation among the parameters $(\chi(\sigma), \mathfrak{f}_\sigma) \in \widehat{\mathbb{Z}}^\times \times \widehat{F}_2'$. Specifically, consider the Dehn twists $x_{1,2}$, $x_{2,3}$, $x_{3,4}$, $x_{4,5}$, and $x_{5,1}$ generating the profinite mapping class group $\widehat{\Gamma}_{0,5}$ of a genus 0 curve with 5 ordered marked points.



Any two such Dehn twists generates a profinite subgroup of $\widehat{\Gamma}_{0,5}$; if these Dehn twists are disjoint they generate the profinite group $\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$, whereas if these Dehn twists intersect they generate the free profinite group $\widehat{F}_2$. Indeed in the latter case we obtain the forgetful morphism $\widehat{\Gamma}_{0,5} \to \widehat{\Gamma}_{0,4}$ sending two such intersecting Dehn twists to x and y respectively, which corresponds to forgetting a marked point (specifically, cutting off one of the pairs of pants at the Dehn twist not intersecting the chosen two). Consequently for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have Galois parameters $\mathfrak{f}_\sigma(x_{1,2}, x_{2,3}), \mathfrak{f}_\sigma(x_{3,4}, x_{4,5}), \mathfrak{f}_\sigma(x_{5,1}, x_{1,2}), \mathfrak{f}_\sigma(x_{2,3}, x_{3,4}), \mathfrak{f}_\sigma(x_{4,5}, x_{5,1}) \in [\widehat{\Gamma}_{0,5}, \widehat{\Gamma}_{0,5}]$, which satisfy a 5-cycle relation due to [Ihara].

Theorem 4.13. *(Ihara) For $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the parameters $(\chi(\sigma), \mathfrak{f}_\sigma) \in \widehat{\mathbb{Z}}^\times \times [\widehat{F}_2, \widehat{F}_2]$ satisfy the profinite 5-cycle relation in $\widehat{\Gamma}_{0,5}$*

$$\mathfrak{f}_\sigma(x_{1,2}, x_{2,3})\mathfrak{f}_\sigma(x_{3,4}, x_{4,5})\mathfrak{f}_\sigma(x_{5,1}, x_{1,2})\mathfrak{f}_\sigma(x_{2,3}, x_{3,4})\mathfrak{f}_\sigma(x_{4,5}, x_{5,1}) = 1$$

The Ihara 5-cycle relation defines a further subgroup $\widehat{\mathbf{GT}} \hookrightarrow \widehat{\mathbf{GT}}_0$ called the Grothendieck-Teichmüller group. For the same reason as in the case $\widehat{\mathbf{GT}}_0$ we have a canonical embedding $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}}$.

Definition 4.3. *The Grothendieck-Teichmüller group $\widehat{\mathbf{GT}}$ is the set of pairs $\sigma = (\lambda, f) \in \widehat{\mathbb{Z}}^\times \times [\widehat{F}_2, \widehat{F}_2]$ acting on $\widehat{F}_2 = \langle x, y\rangle^\wedge$ as $\sigma \cdot x = x^\lambda$ and $\sigma \cdot y = f^{-1} y^\lambda f$, satisfying the profinite relations in $\widehat{\Gamma}_{0,4} = \widehat{F}_2$ and $\widehat{\Gamma}_{0,5}$ respectively*

$$\begin{cases} f(y,x)f(x,y) = 1 & (i) \\ f(z,x)z^{\frac{\lambda-1}{2}} f(y,z) y^{\frac{\lambda-1}{2}} f(x,y) x^{\frac{\lambda-1}{2}} = 1 & (ii) \\ f(x_{1,2},x_{2,3})f(x_{3,4},x_{4,5})f(x_{5,1},x_{1,2})f(x_{2,3},x_{3,4})f(x_{4,5},x_{5,1}) = 1 & (iii) \end{cases}$$

With group operation on $(\lambda, f), (\mu, g) \in \widehat{\mathbf{GT}}$ given by

$$(\lambda, f) \circ (\mu, g) = \big(\lambda\mu, f(x^\mu, g^{-1} y^\mu g)\big)$$

Corollary 4.14. *We have an injection of profinite groups $\iota : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}}$ such that the composition $\chi : \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}} \to \widehat{\mathbb{Z}}^\times$ is the cyclotomic character.*

The surprising result is that the three relations that define $\widehat{\mathbf{GT}}$ completely exhaust the relations coming from the Galois representations $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\pi}_1(\mathcal{M}_{g,n/\overline{\mathbb{Q}}})$, at least in genus 0. Namely, the 2-cycle and 3-cycle relations are the necessary and sufficient conditions in order to define a group which acts on $\widehat{\pi}_1(\mathcal{M}_{0,4/\overline{\mathbb{Q}}})$, while the 5-cycle relation is the necessary and sufficient condition in order to define a further subgroup which acts on $\widehat{\pi}_1(\mathcal{M}_{0,5/\overline{\mathbb{Q}}}, \vec{v})$; from these three relations the action on each $\widehat{\pi}_1(\mathcal{M}_{g,n/\overline{\mathbb{Q}}}, \vec{v})$ is fixed, following [Nakamura-Schneps].

Namely, consider the subgroup $\mathbf{Out}^\flat(\widehat{\Gamma}_{0,n}) \subseteq \mathbf{Out}(\widehat{\Gamma}_{0,n})$ on those outer automorphisms $\sigma \in \mathbf{Out}(\widehat{\Gamma}_{0,n})$ which preserve intertia, that is those outer automorphisms which send conjugacy classes of generators of inertial groups of the normal crossing divisors $\partial\mathcal{M}_{0,n}$ to same conjugacy class up to some power by an element $\chi(\sigma) \in \widehat{\mathbb{Z}}^\times$ which does not depend on the component of $\partial\mathcal{M}_{0,n}$; we denote by $\chi : \mathbf{Out}^\flat(\widehat{\Gamma}_{0,n}) \to \widehat{\mathbb{Z}}^\times$ for the obvious map. Moreover consider the subgroup $\widehat{\mathbf{GT}}_{0,n} = \mathbf{Out}^\flat(\widehat{\Gamma}_{0,n})^{S_n} \subseteq \mathbf{Out}^\flat(\widehat{\Gamma}_{0,n})$ on those outer automorphisms which are S_n-equivariant in that they commute with the canonical action of S_n on $\mathcal{M}_{0,n}$ by permuting marked points.

Theorem 4.15. *(Harbater-Schneps) For $n \geq 4$ the S_n-equivariant morphism $p_n : \mathcal{M}_{0,n+1} \to \mathcal{M}_{0,n}$ induces canonical morphisms $q_n : \mathbf{Out}^\flat(\widehat{\Gamma}_{0,n+1}) \to \mathbf{Out}^\flat(\widehat{\Gamma}_{0,n})$ and $q_n : \widehat{\mathbf{GT}}(n+1) \to \widehat{\mathbf{GT}}(n)$ compatible with inertia $\chi : \mathbf{Out}^\flat(\widehat{\Gamma}_{0,n}) \to \widehat{\mathbb{Z}}^\times$, such that for $n = 4, 5$ we have isomorphisms $\widehat{\mathbf{GT}}_{0,4} \simeq \widehat{\mathbf{GT}}_0$ and $\widehat{\mathbf{GT}}_{0,5} \simeq \widehat{\mathbf{GT}}$, and such that for $n \geq 5$ we have isomorphisms $q_n : \widehat{\mathbf{GT}}_{0,n+1} \xrightarrow{\sim} \widehat{\mathbf{GT}}_{0,n}$ yielding a commutative diagram*

$$\begin{array}{ccccc} \widehat{\mathbf{GT}}_{0,n\geq 5} & \xrightarrow{\sim} & \widehat{\mathbf{GT}}_{0,5} & \xrightarrow{\sim} & \widehat{\mathbf{GT}} \\ & & \downarrow q & & \downarrow q \\ & & \widehat{\mathbf{GT}}_{0,4} & \xrightarrow{\sim} & \widehat{\mathbf{GT}}_0 \end{array}$$

That $\widehat{\mathbf{GT}}$ is an automorphism group of each $\widehat{\pi}_1(\mathcal{M}_{g,n/\overline{\mathbb{Q}}}, \vec{v})$ yields in aninjection $\widehat{\mathbf{GT}} \hookrightarrow \mathbf{Aut}(\widehat{\mathfrak{T}}_0)$ where $\widehat{\mathfrak{T}}_0$ is the tower of étale fundamental groups $\widehat{\pi}_1(\mathcal{M}_{0,n/\overline{\mathbb{Q}}}, \vec{v}) \simeq \widehat{\Gamma}_{0,n}$ for $n \geq 4$ with forgetful morphisms $\widehat{\Gamma}_{0,n+1} \to \widehat{\Gamma}_{0,n}$ between them. The result originally from [Drinfeld] and proved more directly by [Lochak-Schneps] is that this injection is already an isomorphism in genus 0, and is determined by the first two levels, namely by $\mathcal{M}_{0,4}$ of modular dimension 1 and $\mathcal{M}_{0,5}$ of modular dimension 2:

Corollary 4.16. *We have an isomorphism* $\widehat{\mathbf{GT}} \simeq \mathbf{Aut}(\widehat{\mathfrak{T}}_0)$.

There is also a higher genus analog of the above result again due to [Lochak-Schneps], although it is much harder to state. Roughly speaking, the higher genus situation involves the full Teichmüller tower $\widehat{\mathfrak{T}}$ of étale fundamental groupoids $\widehat{\pi}_1(\mathcal{M}_{g,n/\overline{\mathbb{Q}}}, \mathcal{B}_{g,n})$ for $3g-3+n$ with $\mathcal{B}_{g,n}$ the full set of tangential basepoints of $\mathcal{M}_{g,n/\overline{\mathbb{Q}}}$ corresponding to maximal degenerations of type (g,n), along with the forgetful morphisms $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ along with certain gluing morphisms $\mathcal{M}_{g,n} \times \mathcal{M}_{h,m} \to \partial\mathcal{M}_{g+h,n+m-2}$.

Comparatively little is known about the profinite Grothendieck-Teichmüller group $\widehat{\mathbf{GT}}$ and its group theoretic properties. For instance we cannot explicitly write down any elements of $\widehat{\mathbf{GT}}$ except for the obvious elements $(1,1)$ and $(-1,1)$, as is the case for $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the obvious elements corresponding to the identity and complex conjugation. On the other hand, all the known properties of $\widehat{\mathbf{GT}}$ support the following conjecture:

Conjecture. *(Grothendieck) The canonical injection* $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}}$ *is an isomorphism.*

To the extent that $\widehat{\mathbf{GT}}$ is a subgroup of $\mathbf{Aut}(\widehat{F}_2)$ we have an action of $\widehat{\mathbf{GT}}$ on finite étale covers of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. By Belyi's theorem such a finite étale cover is equivalently given by an integral proper normal curve X over an algebraically closed field k of characteristic 0 along with a morphism $\beta : X \to \mathbb{P}^1_k$ ramified only over $\{0,1,\infty\}$. Now if the main conjecture on $\widehat{\mathbf{GT}}$ were true the geometric action of $\widehat{\mathbf{GT}}$ and the arithmetic action of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ should coincide.

Of course if the main conjecture on $\widehat{\mathbf{GT}}$ were true then there should be a well-defined action of $\widehat{\mathbf{GT}}$ on the extension $\overline{\mathbb{Q}}$ fixing $\mathbb{Q}$. Even this is surprisingly hard to show:

Conjecture. *Let* $\beta : X \to \mathbb{P}^1$ *be a Belyi morphism, let* $\sigma \in \widehat{\mathbf{GT}}$*, and let* $\sigma \cdot \beta : \sigma \cdot X \to \mathbb{P}^1$ *be the Belyi morphism given by the action of* σ *on* β*. Then the curve* $\sigma \cdot X$ *is independent of the choice of* β*.*

If this conjecture holds then we immediately obtain a well-defined action of $\widehat{\mathbf{GT}}$ on $\overline{\mathbb{Q}}$ fixing $\mathbb{Q}$ given by its action on elliptic curves over number fields and their j-invariants.

Corollary 4.17. *We have an action of* $\widehat{\mathbf{GT}}$ *on* $\overline{\mathbb{Q}}$ *fixing* $\mathbb{Q}$ *whose restriction to* $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}}$ *is the usual Galois action.*

Proof. Let $\beta : X \to \mathbb{P}^1$ be a Belyi morphism of genus 1, so that the isomorphism class of X over $\overline{\mathbb{Q}}$ is determined by its j-invariant $j(X) \in \overline{\mathbb{Q}}$. Then for $\sigma \in \widehat{\mathbf{GT}}$ the curve $\sigma \cdot X$ has j-invariant $j(\sigma \cdot X) \in \overline{\mathbb{Q}}$, and by conjecture $\sigma \cdot X$ does not depend on the Belyi morphism $\beta : X \to \mathbb{P}^1$ so we obtain a well-defined action $\sigma \cdot j(X) = j(\sigma \cdot X)$ of $\widehat{\mathbf{GT}}$ on $\overline{\mathbb{Q}}$. Moreover for $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathbf{GT}}$ the action coincides with the usual Galois action $\sigma \cdot j(X) = j(\sigma \cdot X)$ on j-invariants and the result follows. □

Meanwhile there are other conjectures having to do with the group theoretic structure of $\widehat{\mathbf{GT}}$, largely inspired by the identical conjectures for $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. One obvious conjecture has to do with the inverse Galois theory of $\widehat{\mathbf{GT}}$. Both $\widehat{\mathbf{GT}}_0$ and $\widehat{\mathbf{GT}}$ are profinite groups, namely the inverse limits of the images of $\widehat{\mathbf{GT}}_0$ and $\widehat{\mathbf{GT}}$ in the finite quotients $\widehat{F}_2/N$ as N varies over the inverse system of open normal subgroups of $\widehat{F}_2$. It is conjectured every finite group occurs as a quotient of $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and analogously we conjecture:

Conjecture. *Every finite group occurs as a quotient of* $\widehat{\mathbf{GT}}$.

Another interesting conjecture involves the derived subgroup of $\widehat{\mathbf{GT}}$. Let $\widehat{\mathbf{GT}}_1$ denote the subgroup on those $(\lambda, f) \in \widehat{\mathbf{GT}}$ with $\lambda = 1$, and let $[\widehat{\mathbf{GT}}, \widehat{\mathbf{GT}}]$ be the commutator subgroup of $\widehat{\mathbf{GT}}$. Since $\mathbf{Gal}(\mathbb{Q}^{\mathbf{ab}}/\mathbb{Q}) \simeq \widehat{\mathbb{Z}}^\times$ we would like to think of $\widehat{\mathbb{Z}}^\times$ as the maximal Abelian quotient of $\widehat{\mathbf{GT}}$ and of $\widehat{\mathbf{GT}}_1$ as the derived quotient of $\widehat{\mathbf{GT}}$. This leads to the following conjecture:

Conjecture. *(Ihara) We have an isomorphism* $\widehat{\mathbf{GT}}_1 \simeq [\widehat{\mathbf{GT}}, \widehat{\mathbf{GT}}]$.

Assuming Ihara's conjecture there is an evident generalization of the Shafarevich conjecture, that $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{\mathbf{ab}}) \simeq [\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$ is a free profinite group of countable rank. In view of Ihara's conjecture on $\widehat{\mathbf{GT}}_1$, the analogous conjecture reads:

Conjecture. *(Ihara-Shafarevich)* $\widehat{\mathbf{GT}}_1 \simeq [\widehat{\mathbf{GT}}, \widehat{\mathbf{GT}}]$ *is a free profinite group.*

On the other hand there are some things that we know about the structure of $\widehat{\mathbf{GT}}$, all of which support the main conjecture. The first has to do with local versions of the Grothendieck-Teichmüller gorup owing to [Andre] using results about the $\mathbb{C}$-analytic fundamental group and the temptered Berkovich $\mathbb{C}_p$-analytic fundamental group. After all, the absolute Galois group $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ should not be considered as a profinite group on its own, but with the additional structure consisting of the local Galois groups $\mathbf{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathbf{Gal}(\overline{\mathbb{Q}}_\infty/\mathbb{Q}_\infty) = \mathbf{Gal}(\mathbb{C}/\mathbb{R})$.

In the Archimedean case the local Grothendieck-Teichmuller group $\widehat{\mathbf{GT}}_\infty$ is given by the closure of $\mathbf{Out}(\Gamma_{0,4}) \simeq \mathbf{GL}_2(\mathbb{Z})$ in $\mathbf{Out}(\widehat{\Gamma}_{0,4})$. Specifically, the canonical inclusion $\Gamma_{0,4} = \pi_1(\mathcal{M}_{0,4}^{\mathbf{an}}, \vec{v}) \subseteq \widehat{\pi}_1(\mathcal{M}_{0,4/\overline{\mathbb{Q}}}, \vec{v}) = \widehat{\Gamma}_{0,4}$ induces an outer Galois representation [4] $\mathbf{Gal}(\mathbb{C}/\mathbb{R}) \to \mathbf{Out}(\Gamma_{0,4}) = \mathbf{GL}_2(\mathbb{Z})$, and for $\widehat{\mathbf{GT}}_\infty = \overline{\mathbf{Out}(\Gamma_{0,4})}$ the closure of $\mathbf{Out}(\Gamma_{0,4})$ in the profinite completion $\mathbf{Out}(\widehat{\Gamma}_{0,4})$, we have the following:

Theorem 4.18. *(André) We have* $\mathbf{Gal}(\mathbb{C}/\mathbb{R}) = \widehat{\mathbf{GT}}_\infty \cap \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \subseteq \widehat{\mathbf{GT}}$.

There is also an analog of the above theorem for the non-Archimedean places of $\mathbb{Q}$. Indeed each $\sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \mathbf{Aut}(F_2^{(p)})$ is determined by parameters $(\chi(\sigma), \mathfrak{f}_\sigma) \in \mathbb{Z}_p^\times \times [F_2^{(p)}, F_2^{(p)}]$ acting on the pro-p-loops $x, y \in \widehat{\pi}_1(\mathcal{M}_{0,4/\overline{\mathbb{Q}}}, \vec{v})^{(p)} \simeq \widehat{F}_2^{(p)}$ as above, and we can define the p-local Grothendieck-Teichmüller group $\mathbf{GT}_p$ using the same pro-p-relations. By André's characterization of the tempered Berkovich $\mathbb{C}_p$-analytic fundamental group we have:

Theorem 4.19. *(André) We have* $\mathbf{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) = \widehat{\mathbf{GT}}_p \cap \mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \subseteq \widehat{\mathbf{GT}}$.

Of course these results are far from the local main conjectures that $\mathbf{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \simeq \widehat{\mathbf{GT}}_p$ and that $\mathbf{Gal}(\mathbb{C}/\mathbb{R}) \simeq \widehat{\mathbf{GT}}_\infty$, but the above theorems by André show that the intersections of the local Grothendieck-Teichmüller groups with the absolute Galois group are not too small.

[4] The Abelianization morphism $F_n \to \mathbb{Z}^n$ induces a morphism $\mathbf{Out}(F_n) \to \mathbf{Out}(\mathbb{Z}^n) \simeq \mathbf{GL}_n(\mathbb{Z})$ with kernel the Torelli group $\mathbf{Tor}(F_n)$. In the case $n = 2$ the morphism $\mathbf{Out}(\Gamma_{0,4}) \to \mathbf{GL}_2(\mathbb{Z})$ is an isomorphism, and the Galois representation $\mathbf{Gal}(\mathbb{C}/\mathbb{R}) \to \mathbf{GL}_2(\mathbb{Z})$ acts by transposition.

In fact for $\mathcal{V}$ an appropriate category of schemes of stacks, we can form its tower of étale fundamental groups and consider its automorphism group $\widehat{\mathbf{GT}}_{\mathcal{V}} = \mathbf{Aut}(\widehat{\mathcal{V}})$. For certain choices of $\mathcal{V}$, a result of [Pop] shows that we already have an isomorphism $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. One particular case from [Pop] settling the Ihara-Oda-Matsumoto conjecture is as follows:

Theorem 4.20. *(Ihara-Oda-Matsumoto, Pop) Let $\mathcal{V}$ be the category of smooth varieties over $\mathbb{Q}$. Then the canonical morphism $\mathbf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbf{GT}}_{\mathcal{V}}$ is an isomorphism.*

There are also local versions of the above, again due to [André]. Moreover following [Pop] we can get by with less; for instance the above morphism is still an isomorphism for certain (large) categories of hyperplane sections. Indeed the main conjecture on $\widehat{\mathbf{GT}}$ amounts to showing that the category $\mathcal{M}$ of moduli spaces $\mathcal{M}_{g,n}$ with the appropriate forgetful and gluing morphisms between them is enough. This remains to be seen.

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