Winter 2018 Probability Qualifying Exam

December 6, 2017

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 2 pages including this page. Make sure you have the right number of pages.
3. Unless otherwise noted all conventions and notation follow that of Çinlar. If you are confused about notation, ask!
4. If necessary you may use the back of pages.

Name:________________________________

UO ID:________________________________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	10	10	10	10	10	90
Score:										

[10 pts] 1. Let $\mathcal{F} = \{\mathcal{F}_n : n \in \mathbb{N}\}$ be a filtration and set $\mathcal{F}_\infty = \bigvee \mathcal{F}_n$. Show that for each bounded random variable $V \in \mathcal{F}_\infty$ there is a sequence of bounded random variables $V_n \in \mathcal{F}_n$ such that V_n converges to V in L^1.

[10 pts] 2. Let (X_n) be a sequence of random variables with positive, finite variance. Show that

$$\limsup_n \frac{X_1 + \cdots + X_n}{\sqrt{n}} = +\infty \quad \text{a.s.}$$

[10 pts] 3. Give an example of σ-algebras $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ (all sub σ-algebras of $\mathcal{H}$) so that $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent, but not conditionally independent given $\mathcal{F}$. Justify.

[10 pts] 4. Suppose (X_n) is a sequence of real-valued random variables which converges in probability. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function show that $f(X_n)$ also converges in probability.

[10 pts] 5. Suppose $(X_n)_{n\in\mathbb{N}}$ is adapted to the filtration $\mathcal{F} = (\mathcal{F}_n)$, each X_n is integrable and define

$$\widehat{X}_n = \frac{1}{n+1}(X_0 + \cdots + X_n)$$

Show that if $\mathbb{E}_n[X_{n+1}] = \widehat{X}_n$, then $\widehat{X}_n$ is an $\mathcal{F}$-martingale.

[10 pts] 6. Suppose $X = (X_n)$ is adapted and integrable. Show that there exists a martingale M and a predictable process A such that $M_0 = A_0 = 0$ and

$$X_n = X_0 + M_n + A_n.$$

Show that this decomposition is unique up to equivalence (that is M_n and A_n are unique almost surely).

[10 pts] 7. Suppose $X = (X_n)$ is a L^1-bounded martingale. Write $X_n = X_n^+ - X_n^-$ where X_n^+ and X_n^- are non-negative. Show that $X^+ = (X_n^+)$ is a submartingale and that

$$Y_n = \lim_m \mathbb{E}_n[X_{n+m}^+]$$

defines a positive L^1-bounded martingale Y.

[10 pts] 8. Fix $s > 1$. Suppose $\{N_p : p \text{ prime}\}$ is an independency of non-negative integer valued random variables with $\mathbb{P}\{N_p = n\} = \frac{p^{-ns}}{(1-p^{-s})}$. Show that

$$\prod_p p^{N_p}$$

is almost surely finite.

[10 pts] 9. Let X be a real-valued random variable with mean μ and finite variance σ^2. Show that

$$\mathbb{P}\{X - \mu \geq x) \leq \frac{\sigma^2}{\sigma^2 + x^2}$$

End of Exam